

GRAPH-LIKE ASYMPTOTICS FOR THE DIRICHLET LAPLACIAN IN CONNECTED TUBULAR DOMAINS

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ABSTRACT. We consider the Dirichlet Laplacian in a waveguide of uniform width and infinite length which is ideally divided into three parts: a “vertex region”, compactly supported and with non zero curvature, and two “edge regions” which are semi-infinite straight strips. We make the waveguide collapse onto a graph by squeezing the edge regions to half-lines and the vertex region to a point. In a setting in which the ratio between the width of the waveguide and the longitudinal extension of the vertex region goes to zero, we prove the convergence of the operator to a selfadjoint realization of the Laplacian on a two edged graph. In the limit operator, the boundary conditions in the vertex depend on the spectral properties of an effective one dimensional Hamiltonian associated to the vertex region.

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1. INTRODUCTION

Metric graphs are objects of great interest as simple but highly not trivial tools to investigate questions covering several topical issues in pure and applied mathematics. They offer a twofold advantage: they are quite simple objects (in many cases formulas for the relevant quantities can be explicitly computed), showing at the same time non trivial features. We evade the duty of giving an exhaustive list of applications of metric graphs and limit ourselves to point out two recent volumes on the subject [3, 12], where an extensive list of references can be found.

Our work is addressed to investigate a long standing problem within the theory of metric graphs, which is finding a rigorous justification for the use of graph models to approximate dynamics in networks of thin waveguides. In many applications the problem is reduced to compare, in some suitable sense, the Laplacian in a squeezing network with a selfadjoint realization of the Laplacian on the graph. A difficult feature of the problem originates from the fact that requiring selfadjointness does not fix univocally the definition of the operator on the graph. Each selfadjoint realization of the Laplacian on a graph is associated to certain boundary (or matching) conditions in the vertices. There are different ways to express the most general selfadjoint boundary conditions, see, e.g., [8] and [25], for the sake of simplicity let us specify one of them for the case of a star-graph \mathcal{G} , that is a graph with N edges of infinite length and one vertex v . We denote by $\mathbb{H}_{\mathcal{G}}$ the Hilbert space naturally associated to \mathcal{G} , a function $x \in \mathbb{H}_{\mathcal{G}}$ is a vector with N components, $x = (x_1, \dots, x_N)$, such that $x_j \in L^2((0, \infty))$. In each edge the vertex v is identified with the origin. Consider an orthogonal projector Π acting in \mathbb{C}^N , and a selfadjoint operator Θ in $\text{Ran}(\Pi^\perp) \subseteq \mathbb{C}^N$, with $\Pi^\perp = 1 - \Pi$. We denote by $-\Delta_{\mathcal{G}}^{\Pi, \Theta}$ the selfadjoint realization of the Laplacian in $\mathbb{H}_{\mathcal{G}}$ associated to the couple of operators (Π, Θ) . Let $x = (x_1, \dots, x_N)$ be a vector in the domain of $-\Delta_{\mathcal{G}}^{\Pi, \Theta}$, then $x_j \in H^2((0, \infty))$ and x must satisfy the boundary conditions

$$\Pi x(v) = 0, \quad \Pi^\perp x'(v) + \Theta \Pi^\perp x(v) = 0,$$

where we denoted by $x(v)$ and $x'(v)$ the vectors in \mathbb{C}^N defined by $x(v) \equiv (x_1(0), \dots, x_N(0))$ and $x'(v) \equiv (x'_1(0), \dots, x'_N(0))$, moreover $-\Delta_{\mathcal{G}}^{\Pi, \Theta} x = (-x''_1, \dots, -x''_N)$.

Among the possible realizations of $-\Delta_{\mathcal{G}}^{\Pi, \Theta}$ we mention:

- The Dirichlet (or decoupling) Laplacian: it is defined by the boundary condition $x(v) = 0$, which corresponds to the choice $\Pi = 1$, in this case $\dim[\text{Ran}(\Pi^\perp)] = 0$.
- The weighted Kirchhoff Laplacian: it is defined by the conditions $\alpha_j x_i(0) = \alpha_i x_j(0)$, for all $i \neq j$, and $\sum_{i=1}^N \bar{\alpha}_i x'_i(0) = 0$, with $\alpha_i \in \mathbb{C}$. This conditions correspond to the choice

$$(1.1) \quad (\Pi)_{ij} = \delta_{ij} - \frac{\alpha_i \bar{\alpha}_j}{\sum_{k=1}^N |\alpha_k|^2}, \quad i, j = 1, \dots, N, \quad \Theta = 0,$$

where δ_{ij} denotes the Kronecker symbol. In this case $\dim[\text{Ran}(\Pi^\perp)] = 1$. When all the constants α_j are equal the operator defined by the choice (1.1) is the so called Kirchhoff (or standard) Laplacian and the boundary conditions read $x_1(0) = \dots = x_N(0)$ and $\sum_{i=1}^N x'_i(0) = 0$.

A central problem in using operators in the family $-\Delta_{\mathcal{G}}^{\Pi, \Theta}$ to approximate the dynamics generated by the Laplacian in a network of thin tubes squeezing to a graph is to understand which boundary conditions in the vertex arise in limit (see, e.g., [13] for a review on this topic). It comes out that this problem strongly depends on what kind of Laplacian is taken in the squeezing network.

With Neumann conditions on the boundary of the squeezing network (or if one considers a network without boundary see, e.g., [14, 30]) the problem is in some sense easier. Under general assumptions on the network properties (see the discussion in [14]), the limit operator on the graph is characterized by weighted Kirchhoff conditions in the vertex, with α_i real and the ratio α_i/α_j being related to the relative radius of network's edges.

At the best of our knowledge the first results in a setting of Neumann networks concerned the analysis of the convergence of stochastic processes associated with elliptic operators on the network, see [18] (see also [2], for recent results on the analysis of stochastic processes in networks of thin tubes). In [26, 33, 14] the convergence of eigenvalues was analyzed while in [30] it was proved the convergence of operators in norm resolvent sense (see also [34, 35]). We also mention the work [4] in which the problem is analyzed by using the theory of Gromov-Hausdorff convergence of manifolds and the techniques developed in [27].

In the setting with Neumann boundary conditions we quote also: the work [15] in which the convergence of resonances is analyzed and the work [16] where the presence of a scaled potential supported in the vertex region is taken into account. It is also worth pointing out that there exist works addressed to the analysis of the corresponding nonlinear problem, see, e.g., [31], and references therein, for the case of squeezing tubular domains and [24] in which it is studied the convergence of semilinear elliptic equations.

The case of a Dirichlet network squeezing to a graph is more complicated and not yet completely understood. One main difficulty is related to the fact that as the width of the network squeezes to zero the energy functional diverges as the inverse of the square of the width. Then to get a meaningful limit a renormalization of the energy is needed. Moreover geometrical perturbations can modify substantially the spectral properties of the Laplacian in the network, a well known phenomenon is, for example, the appearing of isolated eigenvalues in curved waveguides, see [17]. Similar problems arise when the confinement is obtained by an holonomic constrain, see, e.g., [11].

In [29] it is shown that in a Dirichlet network, if the volume of the vertex region is small enough, then the boundary conditions in the vertex are of Dirichlet type. Nevertheless it is known that more general assumptions cannot exclude different limits. In particular it is understood that weighted Kirchhoff conditions of the form described above may arise whenever an effective Hamiltonian associated to the vertex region exhibits eigenvalues at the threshold of the transverse energy see, e.g., [20, 21, 28] and the recent work [10] in which these ideas are further exploited and the occurrence of non homogenous terms in the weighted Kirchhoff condition is pointed out.

In a previous work [1] we considered the Laplacian with Dirichlet conditions on the boundary of a planar waveguide of uniform width collapsing onto a two edged graph. We proved that two possible limits can arise:

- In one case, which we call generic, the limit operator on the graph is defined by the Dirichlet condition in the vertex.
- In a second case, which we call non-generic, the limit operator on the graph is defined by a weighted Kirchhoff condition in the vertex. The occurrence of the non-generic case is related to the existence of a zero energy resonance for an effective one dimensional Hamiltonian associated with the squeezing waveguide.

A crucial hypothesis within the model analyzed in [1] is the assumption that exist two scales of energy: a short one associated to the width of the waveguide, δ ; a large one associated to the longitudinal extension of the vertex region, ε , with $\delta < \varepsilon^{5/2}$. This assumption implies that the dynamics in the waveguide is adiabatically separated into a “fast” component associated to the transverse coordinate and a “slow” component associated to the longitudinal coordinate, making possible to analyze the problem in two steps. First the dynamics is reduced to a one dimensional problem by a projection onto the transverse eigenfunctions and taking the limit $\delta \rightarrow 0$ (in norm resolvent sense); then the operator on the graph can be obtained by taking the limit $\varepsilon \rightarrow 0$ (in norm resolvent sense) of the effective one dimensional Hamiltonian. For the analysis of a similar problem in three dimensional waveguides with methods of Gamma-convergence we mention the paper [9], see also [5].

The weak point of this model, and of the approach used in [1], is that it does not offer enough margin for generalizations to settings in which the underlying graph has three or more edges.

Aim of the present work is to give a better understanding of the result obtained in [1] in view of applications to multi-edged graphs. We review the problem separating explicitly the edges and vertex regions in the waveguide. According to the ideas in [10, 20, 21, 28], we reinterpret the existence of the zero energy resonance (which leads to coupling conditions in the vertex) as an eigenvalue at the energy of the transverse modes for an effective Hamiltonian associated to the vertex region. Moreover in the generic case we obtain better estimates (as compared to [1]) for the rate of convergence ($\delta < \varepsilon^{3/2}$, instead of $\delta < \varepsilon^{5/2}$).

Related to the topic of our paper are also several works which concern the approximation of generic matching conditions in the vertex by scaled Schrödinger operators on the graph, see for example [8] and the recent review [13]. We also point out the paper [19] (and references therein) concerning the analysis of the convergence of Schrödinger operators in dimension one to δ' -type point interactions.

Here is a short summary of our paper. In section 2 we define the waveguide as the union of three parts: the edges and vertex regions. Moreover we define the Dirichlet Laplacian in the waveguide. In section 3 we present our main results (Theorem 1 and Theorem 2). Sections 4 and 5 are devoted to the proofs of the main theorems. In section 6 we explain how to interpret the result stated in Theorems 1 and 2 in terms of the convergence (in norm resolvent sense) to an operator on the graph, see Theorem 3. We close the paper with section 7 in which we summarize our results and point out some generalizations.

2. THE MODEL

We consider a smooth waveguide of uniform width $\delta > 0$ embedded in \mathbb{R}^2 . We assume that the waveguide can be decomposed into three parts: two straight edges $E_{1,\delta}$ and $E_{2,\delta}$ and one vertex region $V_{\delta,\varepsilon}$, $\varepsilon > 0$ being a parameter characterizing the “longitudinal extension” of the vertex region. Throughout the paper we shall always assume that $\delta/\varepsilon \leq 1$. Each edge $E_{j,\delta}$, $j = 1, 2$, can be identified with the manifold (E, \mathfrak{h}_δ) where $E := (0, \infty) \times (0, 1)$ and \mathfrak{h}_δ is the metric

$$\mathfrak{h}_\delta := ds^2 + \delta^2 du^2,$$

where $s \in (0, \infty)$ and $u \in (0, 1)$. The vertex region can be identified with the manifold $(V, \mathbf{g}_{\delta, \varepsilon})$ where $V := (-1, 1) \times (0, 1)$ and $\mathbf{g}_{\delta, \varepsilon}$ is the metric

$$\mathbf{g}_{\delta, \varepsilon} := \varepsilon^2 g_{\delta/\varepsilon} ds^2 + \delta^2 du^2$$

with $s \in (-1, 1)$ and $u \in (0, 1)$. The function $g_{\delta/\varepsilon}(s, u)$ is defined by

$$(2.1) \quad g_{\delta/\varepsilon}(s, u) = (1 + u\delta/\varepsilon\gamma(s))^2,$$

where $\gamma(s)$ is a function of s and we assume that $\gamma \in C_0^\infty((-1, 1))$ and $\|\gamma\|_{L^\infty((-1, 1))} < 1$. We notice that this implies that $g_{\delta/\varepsilon} \in C^\infty(V)$ and that for all $0 < \delta \leq \varepsilon \leq 1$, the bounds

$$(2.2) \quad 0 < (1 - \|\gamma\|_{L^\infty((-1, 1))})^2 \leq \|g_{\delta/\varepsilon}\|_{L^\infty(V)} \leq (1 + \|\gamma\|_{L^\infty((-1, 1))})^2 < 4$$

hold true. The waveguide is obtained by identifying the boundary of $E_{1, \delta}$ corresponding to $s = 0$ with the boundary of $V_{\delta, \varepsilon}$ corresponding to $s = -1$ and the boundary of $E_{2, \delta}$ corresponding to $s = 0$ with the boundary of $V_{\delta, \varepsilon}$ corresponding to $s = 1$.

We denote by $\mathbb{H}_{\delta, \varepsilon}$ the complex Hilbert space

$$\mathbb{H}_{\delta, \varepsilon} := L^2(E_{1, \delta}) \oplus L^2(E_{2, \delta}) \oplus L^2(V_{\delta, \varepsilon}),$$

where $L^2(E_{j, \delta}) \equiv L^2(E, \mathbf{h}_\delta)$, $j = 1, 2$, and $L^2(V_{\delta, \varepsilon}) \equiv L^2(V, \mathbf{g}_{\delta, \varepsilon})$. Moreover we denote by a capital Greek letter a vector in $\mathbb{H}_{\delta, \varepsilon}$, $\Psi = (\psi_1, \psi_2, \psi_v)$. Given two vectors $\Phi \equiv (\phi_1, \phi_2, \phi_v) \in \mathbb{H}_{\delta, \varepsilon}$ and $\Psi \equiv (\psi_1, \psi_2, \psi_v) \in \mathbb{H}_{\delta, \varepsilon}$ the scalar product $(\Phi, \Psi)_{\mathbb{H}_{\delta, \varepsilon}}$ and the norm $\|\Psi\|_{\mathbb{H}_{\delta, \varepsilon}}$ read respectively

$$(\Phi, \Psi)_{\mathbb{H}_{\delta, \varepsilon}} = \sum_{k=1,2} \int_0^\infty \int_0^1 \overline{\phi_k} \psi_k \det[\mathbf{h}_\delta]^{1/2} ds du + \int_{-1}^1 \int_0^1 \overline{\phi_v} \psi_v \det[\mathbf{g}_{\delta, \varepsilon}]^{1/2} ds du,$$

$$\|\Psi\|_{\mathbb{H}_{\delta, \varepsilon}} = [(\Psi, \Psi)_{\mathbb{H}_{\delta, \varepsilon}}]^{1/2},$$

with $\bar{}$ denoting the complex conjugation and

$$(2.3) \quad \det[\mathbf{h}_\delta] = \delta^2, \quad \det[\mathbf{g}_{\delta, \varepsilon}] = \delta^2 \varepsilon^2 g_{\delta/\varepsilon}.$$

In $\mathbb{H}_{\delta, \varepsilon}$ we define the sesquilinear form $Q_{\delta, \varepsilon}$

$$\begin{aligned} Q_{\delta, \varepsilon}[\Phi, \Psi] := & \sum_{k=1,2} \int_0^\infty \int_0^1 \left[\frac{\partial \overline{\phi_k}}{\partial s} \frac{\partial \psi_k}{\partial s} + \frac{1}{\delta^2} \frac{\partial \overline{\phi_k}}{\partial u} \frac{\partial \psi_k}{\partial u} \right] \det[\mathbf{h}_\delta]^{1/2} ds du \\ & + \int_{-1}^1 \int_0^1 \left[\frac{1}{\varepsilon^2 g_{\delta/\varepsilon}} \frac{\partial \overline{\phi_v}}{\partial s} \frac{\partial \psi_v}{\partial s} + \frac{1}{\delta^2} \frac{\partial \overline{\phi_v}}{\partial u} \frac{\partial \psi_v}{\partial u} \right] \det[\mathbf{g}_{\delta, \varepsilon}]^{1/2} ds du. \end{aligned}$$

Below we shall replace the term sesquilinear form by quadratic form (this is justified by the polarization theorem, see, e.g., [23]) and we shall also use the notation $Q_{\delta, \varepsilon}[\Psi] \equiv Q_{\delta, \varepsilon}[\Psi, \Psi]$. Let \mathring{C}^∞ be the set

$$(2.4) \quad \begin{aligned} \mathring{C}^\infty := & \{ \Psi = (\psi_1, \psi_2, \psi_v) \mid \psi_1, \psi_2 \in C_0^\infty(\overline{E}), \psi_v \in C^\infty(V); \\ & \psi_1(s, 0) = \psi_1(s, 1) = \psi_2(s, 0) = \psi_2(s, 1) = \psi_v(s, 0) = \psi_v(s, 1) = 0; \\ & [\partial_s^k \psi_1](0, u) = [(-\varepsilon)^{-k} \partial_s^k \psi_v](-1, u); [\partial_s^k \psi_2](0, u) = [\varepsilon^{-k} \partial_s^k \psi_v](1, u), \forall k \in \mathbb{N}_0 \}, \end{aligned}$$

where we denoted by \overline{E} the closure of E . The domain $D(Q_{\delta, \varepsilon})$ of the quadratic form $Q_{\delta, \varepsilon}$ is the closure of \mathring{C}^∞ equipped with the norm

$$\|\Psi\|_{Q_{\delta, \varepsilon}} := (\|\Psi\|_{\mathbb{H}_{\delta, \varepsilon}}^2 + Q_{\delta, \varepsilon}[\Psi])^{1/2}.$$

We denote by $H_{\delta, \varepsilon}$ the unique selfadjoint operator in $\mathbb{H}_{\delta, \varepsilon}$ associated to the quadratic form $Q_{\delta, \varepsilon}$

$$(2.5) \quad D(H_{\delta, \varepsilon}) := \{ \Psi \in D(Q_{\delta, \varepsilon}) \mid \forall \Phi \in D(Q_{\delta, \varepsilon}), Q_{\delta, \varepsilon}[\Phi, \Psi] = \langle \Phi, \Xi \rangle_{\mathbb{H}_{\delta, \varepsilon}}; \Xi \in \mathbb{H}_{\delta, \varepsilon} \}$$

$$(2.6) \quad H_{\delta, \varepsilon} \Psi := \Xi$$

(see, e.g., [23] for the concept of selfadjoint operators associated to closed quadratic forms).

2.1. Unitarily equivalent Hamiltonian. Let $L^2(E)$ and $L^2(V)$ be the complex Hilbert spaces endowed with the norms

$$\|\tilde{\psi}_j\|_{L^2(E)}^2 := \int_0^\infty \int_0^1 |\tilde{\psi}_j|^2 ds du, \quad j = 1, 2; \quad \|\tilde{\psi}_v\|_{L^2(V)}^2 := \int_{-1}^1 \int_0^1 |\tilde{\psi}_j|^2 ds du.$$

Let us denote by $\tilde{\mathbb{H}}_\varepsilon$ the complex Hilbert space

$$\tilde{\mathbb{H}}_\varepsilon := L^2(E) \oplus L^2(E) \oplus L^2(V, \varepsilon ds du).$$

Given two vectors $\tilde{\Phi} \equiv (\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_v) \in \tilde{\mathbb{H}}_\varepsilon$ and $\tilde{\Psi} \equiv (\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_v) \in \tilde{\mathbb{H}}_\varepsilon$ the scalar product $(\tilde{\Phi}, \tilde{\Psi})_{\tilde{\mathbb{H}}_\varepsilon}$ and the norm $\|\tilde{\Psi}\|_{\tilde{\mathbb{H}}_\varepsilon}$ read

$$\begin{aligned} (\tilde{\Phi}, \tilde{\Psi})_{\tilde{\mathbb{H}}_\varepsilon} &= \sum_{k=1,2} \int_0^\infty \int_0^1 \overline{\tilde{\phi}_k} \tilde{\psi}_k ds du + \varepsilon \int_{-1}^1 \int_0^1 \overline{\tilde{\phi}_v} \tilde{\psi}_v ds du, \\ \|\tilde{\Psi}\|_{\tilde{\mathbb{H}}_\varepsilon} &= \left(\|\tilde{\psi}_1\|_{L^2(E)}^2 + \|\tilde{\psi}_2\|_{L^2(E)}^2 + \varepsilon \|\tilde{\psi}_v\|_{L^2(V)}^2 \right)^{1/2}. \end{aligned}$$

For all $0 < \varepsilon \leq 1$ we denote by $U_{\delta,\varepsilon}$ the unitary map $U_{\delta,\varepsilon} : \mathbb{H}_{\delta,\varepsilon} \rightarrow \tilde{\mathbb{H}}_\varepsilon$ defined by

$$\begin{aligned} (\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_v) &\equiv U_{\delta,\varepsilon}(\psi_1, \psi_2, \psi_v) := (\det[\mathfrak{h}_\delta]^{1/4} \psi_1, \det[\mathfrak{h}_\delta]^{1/4} \psi_2, \varepsilon^{-1/2} \det[\mathfrak{g}_{\delta,\varepsilon}]^{1/4} \psi_v) \\ (2.7) \quad &= (\delta^{1/2} \psi_1, \delta^{1/2} \psi_2, \delta^{1/2} g_{\delta/\varepsilon}^{1/4} \psi_v), \end{aligned}$$

where we used equation (2.3).

For all $0 < \delta \leq \varepsilon \leq 1$, we denote by $\mathring{H}_{\delta,\varepsilon}^1$ the closure of \mathring{C}^∞ , defined in equation (2.4), with respect to the norm

$$(2.8) \quad \|\Psi\|_{\mathring{H}_{\delta,\varepsilon}^1}^2 := \|\Psi\|_{\tilde{\mathbb{H}}_\varepsilon}^2 + \sum_{j=1,2} \left[\|\partial_s \tilde{\psi}_j\|_{L^2(E)}^2 + \delta^{-2} \|\partial_u \tilde{\psi}_j\|_{L^2(E)}^2 \right] + \varepsilon \left[\varepsilon^{-2} \|\partial_s \tilde{\psi}_v\|_{L^2(V)}^2 + \delta^{-2} \|\partial_u \tilde{\psi}_v\|_{L^2(V)}^2 \right],$$

and by $\mathring{H}_{\delta,\varepsilon}^2$ the closure of \mathring{C}^∞ with respect to the norm

$$(2.9) \quad \|\Psi\|_{\mathring{H}_{\delta,\varepsilon}^2}^2 := \|\Psi\|_{\mathring{H}_{\delta,\varepsilon}^1}^2 + \sum_{j=1,2} \left[\|\partial_s^2 \tilde{\psi}_j\|_{L^2(E)}^2 + \delta^{-4} \|\partial_u^2 \tilde{\psi}_j\|_{L^2(E)}^2 \right] + \varepsilon \left[\varepsilon^{-4} \|\partial_s^2 \tilde{\psi}_v\|_{L^2(V)}^2 + \delta^{-4} \|\partial_u^2 \tilde{\psi}_v\|_{L^2(V)}^2 \right].$$

We notice that $\mathring{H}_{\delta,\varepsilon}^1$ and $\mathring{H}_{\delta,\varepsilon}^2$ coincide with

$$\begin{aligned} \mathring{H}_{\delta,\varepsilon}^1 &= \{ \tilde{\Psi} \equiv (\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_v) \in \tilde{\mathbb{H}}_\varepsilon \mid \|\tilde{\Psi}\|_{\mathring{H}_{\delta,\varepsilon}^1} < \infty; \\ (2.10) \quad &\tilde{\psi}_1|_{u=0,1} = \tilde{\psi}_2|_{u=0,1} = 0 \text{ in } L^2((0, \infty)); \quad \tilde{\psi}_v|_{u=0,1} = 0 \text{ in } L^2((-1, 1)); \\ &\tilde{\psi}_1|_{s=0} = \tilde{\psi}_v|_{s=-1}, \quad \tilde{\psi}_2|_{s=0} = \tilde{\psi}_v|_{s=1} \text{ in } L^2((0, 1)) \}, \end{aligned}$$

$$\begin{aligned} \mathring{H}_{\delta,\varepsilon}^2 &= \left\{ \tilde{\Psi} \equiv (\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_v) \in \tilde{\mathbb{H}}_\varepsilon \mid \|\tilde{\Psi}\|_{\mathring{H}_{\delta,\varepsilon}^2} < \infty; \right. \\ (2.11) \quad &\tilde{\psi}_1|_{u=0,1} = \tilde{\psi}_2|_{u=0,1} = 0 \text{ in } L^2((0, \infty)); \quad \tilde{\psi}_v|_{u=0,1} = 0 \text{ in } L^2((-1, 1)); \\ &\tilde{\psi}_1|_{s=0} = \tilde{\psi}_v|_{s=-1}; \quad \tilde{\psi}_2|_{s=0} = \tilde{\psi}_v|_{s=1} \text{ in } L^2((0, 1)); \\ &\left. \frac{\partial \tilde{\psi}_1}{\partial s} \Big|_{s=0} = -\frac{1}{\varepsilon} \frac{\partial \tilde{\psi}_v}{\partial s} \Big|_{s=-1}; \quad \frac{\partial \tilde{\psi}_2}{\partial s} \Big|_{s=0} = \frac{1}{\varepsilon} \frac{\partial \tilde{\psi}_v}{\partial s} \Big|_{s=1} \text{ in } L^2((0, 1)) \right\}. \end{aligned}$$

In formulae (2.10) and (2.11), in the boundary values, the symbols $|_{u=0,1}$ and $|_{s=0,\pm 1}$ must be intended as trace operators, see, e.g., [32]. We recall that by standard Sobolev trace theorems, see, e.g., [32], for any $0 < \delta \leq \varepsilon \leq 1$ and $p = 1, 2$, $\tilde{\Psi} \in \mathring{H}_{\delta,\varepsilon}^p$ implies $\tilde{\psi}_j|_{s=0}, \tilde{\psi}_v|_{s=\pm 1} \in H^{p-1/2}((0, 1))$, $\tilde{\psi}_j|_{u=0,1} \in H^{p-1/2}((0, \infty))$ and $\tilde{\psi}_v|_{u=0,1} \in H^{p-1/2}((-1, 1))$, with $j = 1, 2$. Moreover $\tilde{\Psi} \in \mathring{H}_{\delta,\varepsilon}^2$ implies $\partial_s \tilde{\psi}_j|_{s=0}, \partial_s \tilde{\psi}_v|_{s=\pm 1} \in H^{1/2}((0, 1))$, with $j = 1, 2$.

For $p = 1, 2$, and $0 < a, b \leq \infty$, we shall denote by $H^p((a, b))$ the standard Sobolev space of order p associated to the interval (a, b) , see, e.g., [32].

Proposition 2.1. *The Hamiltonian $H_{\delta, \varepsilon}$ is unitarily equivalent to the Hamiltonian $\tilde{H}_{\delta, \varepsilon} : D(\tilde{H}_{\delta, \varepsilon}) \subset \tilde{\mathbb{H}}_\varepsilon \rightarrow \tilde{\mathbb{H}}_\varepsilon$ defined by*

$$(2.12) \quad D(\tilde{H}_{\delta, \varepsilon}) = \dot{H}_{\delta, \varepsilon}^2$$

$$(2.13) \quad \tilde{H}_{\delta, \varepsilon}(\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_v) = \left(\left[-\frac{\partial^2}{\partial s^2} - \frac{1}{\delta^2} \frac{\partial^2}{\partial u^2} \right] \tilde{\psi}_1, \left[-\frac{\partial^2}{\partial s^2} - \frac{1}{\delta^2} \frac{\partial^2}{\partial u^2} \right] \tilde{\psi}_2, \frac{1}{\varepsilon^2} \tilde{L}_{\delta/\varepsilon} \tilde{\psi}_v \right),$$

where $\tilde{L}_{\delta/\varepsilon}$ denotes the linear operator

$$(2.14) \quad \tilde{L}_{\delta/\varepsilon} := -\frac{1}{g_{\delta/\varepsilon}} \frac{\partial^2}{\partial s^2} - \left[\frac{\partial}{\partial s} \frac{1}{g_{\delta/\varepsilon}} \right] \frac{\partial}{\partial s} + W_{\delta/\varepsilon} - \frac{1}{(\delta/\varepsilon)^2} \frac{\partial^2}{\partial u^2}$$

with

$$(2.15) \quad W_{\delta/\varepsilon} = -\frac{1}{4} \frac{\gamma^2(s)}{(1 + u\delta/\varepsilon\gamma(s))^2} + \frac{1}{2} \frac{u\delta/\varepsilon\dot{\gamma}(s)}{(1 + u\delta/\varepsilon\gamma(s))^3} - \frac{5}{4} \frac{(u\delta/\varepsilon\dot{\gamma}(s))^2}{(1 + u\delta/\varepsilon\gamma(s))^4}.$$

Proof. We prove first that the quadratic form $Q_{\delta, \varepsilon}$ in $\mathbb{H}_{\delta, \varepsilon}$ is unitarily equivalent to the quadratic form $\tilde{Q}_{\delta, \varepsilon}$ in $\tilde{\mathbb{H}}_\varepsilon$ defined by

$$\begin{aligned} \tilde{Q}_{\delta, \varepsilon}[\tilde{\Phi}, \tilde{\Psi}] &:= \sum_{k=1,2} \int_0^\infty \int_0^1 \left[\overline{\frac{\partial \tilde{\phi}_k}{\partial s}} \frac{\partial \tilde{\psi}_k}{\partial s} + \frac{1}{\delta^2} \overline{\frac{\partial \tilde{\phi}_k}{\partial u}} \frac{\partial \tilde{\psi}_k}{\partial u} \right] ds du \\ &\quad + \varepsilon \int_{-1}^1 \int_0^1 \frac{1}{\varepsilon^2} \left(\frac{1}{g_{\delta/\varepsilon}} \overline{\frac{\partial \tilde{\phi}_v}{\partial s}} \frac{\partial \tilde{\psi}_v}{\partial s} + \widetilde{W}_{\delta/\varepsilon} \tilde{\phi}_v \tilde{\psi}_v \right) ds du \\ &\quad + \varepsilon \int_{-1}^1 \int_0^1 \frac{1}{\delta^2} \left(\overline{\frac{\partial \tilde{\phi}_v}{\partial u}} \frac{\partial \tilde{\psi}_v}{\partial u} + \widetilde{\widetilde{W}}_{\delta/\varepsilon} \tilde{\phi}_v \tilde{\psi}_v \right) ds du \end{aligned}$$

with $D(\tilde{Q}_{\delta, \varepsilon}) = \dot{H}_{\delta, \varepsilon}^1$ (see formula (2.10) for the definition of $\dot{H}_{\delta, \varepsilon}^1$) and

$$(2.16) \quad \widetilde{W}_{\delta, \varepsilon} = -\frac{\partial}{\partial s} \left[\frac{1}{g_{\delta/\varepsilon}^{3/4}} \frac{\partial}{\partial s} (g_{\delta/\varepsilon}^{-1/4}) \right] + \frac{1}{g_{\delta/\varepsilon}^{1/2}} \left(\frac{\partial}{\partial s} g_{\delta/\varepsilon}^{-1/4} \right)^2,$$

$$(2.17) \quad \widetilde{\widetilde{W}}_{\delta, \varepsilon} = -\frac{\partial}{\partial u} \left[g_{\delta/\varepsilon}^{1/4} \frac{\partial}{\partial u} (g_{\delta/\varepsilon}^{-1/4}) \right] + g_{\delta/\varepsilon}^{1/2} \left(\frac{\partial}{\partial u} g_{\delta/\varepsilon}^{-1/4} \right)^2.$$

Let us denote by $\tilde{Q}_{\delta, \varepsilon}^0$ the restriction of the quadratic form $\tilde{Q}_{\delta, \varepsilon}$ to \dot{C}^∞ , see the definition (2.4). Since $g_{\delta/\varepsilon} \in C^\infty(V)$ and the bounds (2.2) hold true, the norm

$$\|\tilde{\Psi}\|_{\tilde{Q}_{\delta, \varepsilon}} := (\|\tilde{\Psi}\|_{\tilde{\mathbb{H}}_{\delta, \varepsilon}}^2 + \tilde{Q}_{\delta, \varepsilon}[\tilde{\Psi}])^{1/2},$$

is equivalent to the $\dot{H}_{\delta, \varepsilon}^1$ -norm defined in equation (2.8). Then the quadratic form $\tilde{Q}_{\delta, \varepsilon}$ is the closure of $\tilde{Q}_{\delta, \varepsilon}^0$ in the norm $\|\cdot\|_{\tilde{Q}_{\delta, \varepsilon}}$. For any function $\tilde{\Psi} \in \dot{C}^\infty$, see the definition (2.4), one has $\Psi = U_{\delta, \varepsilon}^{-1} \tilde{\Psi} \in \dot{C}^\infty$, where $U_{\delta, \varepsilon}$ is the unitary map defined in (2.7) and $U_{\delta, \varepsilon}^{-1}$ its inverse. Let us define

$$\tilde{Q}_{\delta, \varepsilon}^\#[\tilde{\Phi}, \tilde{\Psi}] := Q_{\delta, \varepsilon}[U_{\delta, \varepsilon}^{-1} \tilde{\Phi}, U_{\delta, \varepsilon}^{-1} \tilde{\Psi}_{\delta, \varepsilon}].$$

From the definitions of $U_{\delta,\varepsilon}$ and $Q_{\delta,\varepsilon}$ we have

$$\begin{aligned}\tilde{Q}_{\delta,\varepsilon}^{\#}[\tilde{\Phi}, \tilde{\Psi}] &= \sum_{k=1,2} \int_0^\infty \int_0^1 \left[\frac{\overline{\partial \tilde{\phi}_k}}{\partial s} \frac{\partial \tilde{\psi}_k}{\partial s} + \frac{1}{\delta^2} \frac{\overline{\partial \tilde{\phi}_k}}{\partial u} \frac{\partial \tilde{\psi}_k}{\partial u} \right] ds du \\ &\quad + \int_{-1}^1 \int_0^1 \left[\frac{1}{\varepsilon^2 g_{\delta/\varepsilon}^{1/2}} \frac{\partial}{\partial s} (g_{\delta/\varepsilon}^{-1/4} \tilde{\phi}_v) \frac{\partial}{\partial s} (g_{\delta/\varepsilon}^{-1/4} \tilde{\psi}_v) + \frac{g_{\delta/\varepsilon}^{1/2}}{\delta^2} \frac{\partial}{\partial u} (g_{\delta/\varepsilon}^{-1/4} \tilde{\phi}_v) \frac{\partial}{\partial u} (g_{\delta/\varepsilon}^{-1/4} \tilde{\psi}_v) \right] \varepsilon ds du.\end{aligned}$$

We notice that

$$\begin{aligned}&\int_{-1}^1 \int_0^1 \left[\frac{1}{\varepsilon^2 g_{\delta/\varepsilon}^{1/2}} \frac{\partial}{\partial s} (g_{\delta/\varepsilon}^{-1/4} \tilde{\phi}_v) \frac{\partial}{\partial s} (g_{\delta/\varepsilon}^{-1/4} \tilde{\psi}_v) \right] \varepsilon ds du = \\ &= \int_{-1}^1 \int_0^1 \frac{1}{\varepsilon^2 g_{\delta/\varepsilon}} \frac{\overline{\partial \tilde{\phi}_v}}{\partial s} \frac{\partial \tilde{\psi}_v}{\partial s} \varepsilon ds du \\ &\quad + \int_{-1}^1 \int_0^1 \left[\frac{1}{\varepsilon^2 g_{\delta/\varepsilon}^{3/4}} \frac{\partial}{\partial s} (g_{\delta/\varepsilon}^{-1/4}) \left(\frac{\overline{\partial \tilde{\phi}_v}}{\partial s} \tilde{\psi}_v + \overline{\tilde{\phi}_v} \frac{\partial \tilde{\psi}_v}{\partial s} \right) + \frac{1}{\varepsilon^2 g_{\delta/\varepsilon}^{1/2}} \left(\frac{\partial}{\partial s} g_{\delta/\varepsilon}^{-1/4} \right)^2 \tilde{\phi}_v \tilde{\psi}_v \right] \varepsilon ds du,\end{aligned}$$

and

$$\begin{aligned}&\int_{-1}^1 \int_0^1 \frac{g_{\delta/\varepsilon}^{1/2}}{\delta^2} \frac{\partial}{\partial u} (g_{\delta/\varepsilon}^{-1/4} \tilde{\phi}_v) \frac{\partial}{\partial u} (g_{\delta/\varepsilon}^{-1/4} \tilde{\psi}_v) \varepsilon ds du \\ &= \int_{-1}^1 \int_0^1 \frac{1}{\delta^2} \frac{\overline{\partial \tilde{\phi}_v}}{\partial u} \frac{\partial \tilde{\psi}_v}{\partial u} \varepsilon ds du \\ &\quad + \int_{-1}^1 \int_0^1 \left[\frac{g_{\delta/\varepsilon}^{1/4}}{\delta^2} \frac{\partial}{\partial u} (g_{\delta/\varepsilon}^{-1/4}) \left(\frac{\overline{\partial \tilde{\phi}_v}}{\partial u} \tilde{\psi}_v + \overline{\tilde{\phi}_v} \frac{\partial \tilde{\psi}_v}{\partial u} \right) + \frac{g_{\delta/\varepsilon}^{1/2}}{\delta^2} \left(\frac{\partial}{\partial u} g_{\delta/\varepsilon}^{-1/4} \right)^2 \tilde{\phi}_v \tilde{\psi}_v \right] \varepsilon ds du.\end{aligned}$$

By integration by parts in s one has

$$\begin{aligned}&\int_{-1}^1 \int_0^1 \left[\frac{1}{\varepsilon^2 g_{\delta/\varepsilon}^{3/4}} \frac{\partial}{\partial s} (g_{\delta/\varepsilon}^{-1/4}) \left(\frac{\overline{\partial \tilde{\phi}_v}}{\partial s} \tilde{\psi}_v + \overline{\tilde{\phi}_v} \frac{\partial \tilde{\psi}_v}{\partial s} \right) + \frac{1}{\varepsilon^2 g_{\delta/\varepsilon}^{1/2}} \left(\frac{\partial}{\partial s} g_{\delta/\varepsilon}^{-1/4} \right)^2 \tilde{\phi}_v \tilde{\psi}_v \right] \varepsilon ds du \\ &= \int_{-1}^1 \int_0^1 \frac{1}{\varepsilon^2} \widetilde{W}_{\delta/\varepsilon} \tilde{\phi}_v \tilde{\psi}_v \varepsilon ds du,\end{aligned}$$

where $\widetilde{W}_{\delta/\varepsilon}$ was defined in equation (2.16) and we used the fact that the boundary terms are null because $\gamma \in C_0^\infty((-1, 1))$. By integration by parts in u one has

$$\begin{aligned}&\int_{-1}^1 \int_0^1 \left[\frac{g_{\delta/\varepsilon}^{1/4}}{\delta^2} \frac{\partial}{\partial u} (g_{\delta/\varepsilon}^{-1/4}) \left(\frac{\overline{\partial \tilde{\phi}_v}}{\partial u} \tilde{\psi}_v + \overline{\tilde{\phi}_v} \frac{\partial \tilde{\psi}_v}{\partial u} \right) + \frac{g_{\delta/\varepsilon}^{1/2}}{\delta^2} \left(\frac{\partial}{\partial u} g_{\delta/\varepsilon}^{-1/4} \right)^2 \tilde{\phi}_v \tilde{\psi}_v \right] \varepsilon ds du \\ &= \int_{-1}^1 \int_0^1 \frac{1}{\delta^2} \widetilde{\widetilde{W}}_{\delta/\varepsilon} \tilde{\phi}_v \tilde{\psi}_v \varepsilon ds du,\end{aligned}$$

where $\widetilde{\widetilde{W}}_{\delta/\varepsilon}$ was defined in equation (2.16) and we used the fact that the boundary terms are null because $\tilde{\phi}_v|_{u=0,1} = \tilde{\psi}_v|_{u=0,1} = 0$. By noticing that $\tilde{Q}_{\delta,\varepsilon}^{\#}[\tilde{\Phi}, \tilde{\Psi}] \equiv \tilde{Q}_{\delta,\varepsilon}[\tilde{\Phi}, \tilde{\Psi}]$ one deduces the unitary equivalence of the quadratic forms $Q_{\delta,\varepsilon}$ and $\tilde{Q}_{\delta,\varepsilon}$.

Let $\tilde{H}_{\delta,\varepsilon}^\#$ the Hamiltonian in $\tilde{\mathbb{H}}_\varepsilon$ associated to the quadratic form $\tilde{Q}_{\delta,\varepsilon}$. By integrating by parts in s and u , one sees that for any $\tilde{\Phi}, \tilde{\Psi} \in D(\tilde{Q}_{\delta,\varepsilon})$

(2.18)

$$\begin{aligned} \tilde{Q}_{\delta,\varepsilon}[\tilde{\Phi}, \tilde{\Psi}] &= \sum_{k=1,2} \int_0^\infty \int_0^1 \overline{\tilde{\phi}_k} \left[-\frac{\partial^2 \tilde{\psi}_k}{\partial s^2} - \frac{1}{\delta^2} \frac{\partial^2 \tilde{\psi}_k}{\partial u^2} \right] ds du \\ &+ \int_{-1}^1 \int_0^1 \overline{\tilde{\phi}_v} \left[\frac{1}{\varepsilon^2} \left(-\frac{1}{g_{\delta/\varepsilon}} \frac{\partial^2 \tilde{\psi}_v}{\partial s^2} - \left(\frac{\partial}{\partial s} \frac{1}{g_{\delta/\varepsilon}} \right) \tilde{\psi}_v + (\widetilde{W}_{\delta/\varepsilon} + (\varepsilon/\delta)^2 \widetilde{\widetilde{W}}_{\delta/\varepsilon}) \tilde{\psi}_v \right) - \frac{1}{\delta^2} \frac{\partial^2 \tilde{\psi}_v}{\partial u^2} \right] \varepsilon ds du \\ &- \int_0^1 \overline{\tilde{\phi}_1} \Big|_{s=0} \left[\frac{\partial \tilde{\psi}_1}{\partial s} \Big|_{s=0} + \frac{1}{\varepsilon} \frac{\partial \tilde{\psi}_v}{\partial s} \Big|_{s=-1} \right] du - \int_0^1 \overline{\tilde{\phi}_2} \Big|_{s=0} \left[\frac{\partial \tilde{\psi}_2}{\partial s} \Big|_{s=0} - \frac{1}{\varepsilon} \frac{\partial \tilde{\psi}_v}{\partial s} \Big|_{s=1} \right] du, \end{aligned}$$

where we used the fact that $g_{\delta/\varepsilon}(\pm 1, u) = 1$ and the fact that $\tilde{\Phi} \in \dot{H}_{\delta,\varepsilon}^1$. A straightforward calculation gives

$$-\left(\frac{\partial}{\partial s} \frac{1}{g_{\delta/\varepsilon}} \right) + \widetilde{W}_{\delta/\varepsilon} + (\varepsilon/\delta)^2 \widetilde{\widetilde{W}}_{\delta/\varepsilon} = W_{\delta/\varepsilon}.$$

Then by the first representation theorem, see, e.g., [23], we have that the Hamiltonian $\tilde{H}_{\delta,\varepsilon}^\#$ acts as $\tilde{H}_{\delta,\varepsilon}$ defined in equation (2.13). Moreover the domain of $\tilde{H}_{\delta,\varepsilon}^\#$ is given by the functions in $\dot{H}_{\delta,\varepsilon}^1$ such that the boundary terms in equation (2.18) are zero and

$$\|\tilde{H}_{\delta,\varepsilon}^\# \tilde{\Psi}\|_{\tilde{\mathbb{H}}_\varepsilon} < \infty.$$

It is easy to convince oneself that since $g_{\delta/\varepsilon} \in C^\infty(V)$ and the bounds (2.2) hold true, one has $D(\tilde{H}_{\delta,\varepsilon}^\#) = \dot{H}_{\delta,\varepsilon}^2$ which implies $\tilde{H}_{\delta,\varepsilon}^\# \equiv \tilde{H}_{\delta,\varepsilon}$. \square

3. APPROXIMATE SOLUTION OF THE RESOLVENT EQUATION

In this section we give an approximate solution to the resolvent equation $(\tilde{H}_{\delta,\varepsilon} - z)\tilde{\Psi} = \tilde{\Xi}$, for $z \in \mathbb{C} \setminus \mathbb{R}$ and for some suitable choice of $\tilde{\Xi} \in \tilde{\mathbb{H}}_\varepsilon$, see Theorem 1 below. In the analysis we are forced to renormalize the spectral parameter z to $z + n^2\pi^2/\delta^2$, with n integer. In the final part of the section we discuss the behavior of the approximate solution as $\varepsilon \rightarrow 0$.

We denote by h_v the Hamiltonian in $L^2((-1, 1))$

$$(3.1) \quad D(h_v) := \{y \in H^2((-1, 1)) \mid y'(\pm 1) = 0\}$$

$$(3.2) \quad h_v := -\frac{d^2}{ds^2} - \frac{\gamma^2(s)}{4}.$$

For any function $y \in L^2((-1, 1))$ we use the notation

$$\|y\|_{v,s} := \|y\|_{L^2((-1,1))}; \quad \|y\|_{v,s,\infty} := \|y\|_{L^\infty((-1,1))}$$

and similarly for the scalar product in $L^2((-1, 1))$. For $n = 1, 2, 3, \dots$ we denote by $y_n(s)$ the (real) eigenfunctions of h_v and by λ_n the corresponding eigenvalues arranged in increasing order

$$h_v y_n = -y_n'' - \frac{\gamma^2}{4} y_n = \lambda_n y_n; \quad y_n'(\pm 1) = 0 \quad n \in \mathbb{N}.$$

We assume the normalization $(y_n, y_m)_{v,s} = \delta_{n,m}$, $n, m \in \mathbb{N}$. For any $z \in \mathbb{C} \setminus \mathbb{R}$ we denote by $r_v(z)$ the resolvent of h_v ,

$$r_v(z) := (h_v - z)^{-1}; \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

For $z \in \mathbb{C} \setminus \mathbb{R}$ the operator $r_v(z)$ is bounded¹, in fact $\|r_v(z)\|_{\mathcal{B}(L^2((-1,1)))} \leq 1/|\operatorname{Im} z|$, and $r_v(z) : L^2((-1, 1)) \rightarrow D(h_v)$. The integral kernel of $r_v(z)$ can be written as

$$(3.3) \quad r_v(z; s, s') = \sum_n \frac{y_n(s) y_n(s')}{\lambda_n - z}.$$

¹By $\mathcal{B}(\cdot)$ we denote the Banach space of bounded linear operators in some Hilbert space.

For n large enough one has $(n - 1/2)^2 \pi^2 / 4 < \lambda_n < (n + 1/2)^2 \pi^2 / 4$ and $\|y_n\|_{v,s,\infty} \leq c$ where c does not depend on n (see, e.g., [36]). Thus the series in (3.3) converges absolutely and pointwise for $s, s' \in [-1, 1]$. As a function of z , the operator $r_v(z)$ can be analytically continued to a linear bounded operator for $z \in \mathbb{C} \setminus \{\lambda_n\}$.

We give a formula for the kernel $r_v(z; s, s')$ which does not involve series (see, e.g., [22, Ch 4.2]). For any $z \in \mathbb{C} \setminus \mathbb{R}$, let $\zeta_v(z)$ and $\eta_v(z)$ be two generic solutions of the equations

$$(3.4) \quad \begin{aligned} -\zeta_v''(z) + (-\gamma^2/4 - z)\zeta_v(z) &= 0; & \zeta_v'(z; -1) &= 0 \\ -\eta_v''(z) + (-\gamma^2/4 - z)\eta_v(z) &= 0; & \eta_v'(z; 1) &= 0 \end{aligned}$$

and let $\mathcal{W}_v(z; s)$ be the Wronskian

$$\mathcal{W}_v(z; s) := \eta_v(z; s)\zeta_v'(z; s) - \zeta_v(z; s)\eta_v'(z; s).$$

We notice that $\mathcal{W}_v(z; s)$ does not depend on s , see, e.g., [22], and we set $\mathcal{W}_v(z) \equiv \mathcal{W}_v(z; s)$. Then the integral kernel of $r_v(z)$ reads

$$(3.5) \quad r_v(z; s, s') = \begin{cases} \frac{\zeta_v(z; s)\eta_v(z; s')}{\mathcal{W}_v(z)} & s \leq s' \\ \frac{\eta_v(z; s)\zeta_v(z; s')}{\mathcal{W}_v(z)} & s > s' \end{cases}$$

We distinguish two cases:

Case 1. Zero is not an eigenvalue of the Hamiltonian h_v defined in (3.1) - (3.2)

Case 2. Zero is an eigenvalue of the Hamiltonian h_v defined in (3.1) - (3.2). Then we denote by n^* the integer corresponding to the eigenvalue zero, i.e., $\lambda_{n^*} = 0$ and by $y^* \equiv y_{n^*}$ the corresponding eigenfunction, i.e.,

$$(3.6) \quad h_v y^* = -\frac{d^2}{ds^2} y^* - \frac{\gamma^2}{4} y^* = 0$$

with $y^{*'}(\pm 1) = 0$ and $\|y^*\|_{v,s} = 1$ (here and below $y^{*'}$ denotes the derivative of y^*). We define the constants

$$(3.7) \quad \alpha_1 := y^*(-1); \quad \alpha_2 := y^*(1).$$

By the definition of h_v we have that if y_n is an eigenfunction of h_v then so is \bar{y}_n . Hence we can assume that y^* is real, which in turns implies $\alpha_1, \alpha_2 \in \mathbb{R}$.

Let us denote by $\chi_n(u)$ the functions

$$(3.8) \quad \chi_n(u) := \sqrt{2} \sin(n\pi u); \quad n \in \mathbb{N}.$$

Definition 3.1. For any vector $\tilde{\Xi}_n \equiv (f_1 \chi_n, f_2 \chi_n, 0)$ with $f_1, f_2 \in L^2((0, \infty))$ and $z \in \mathbb{C} \setminus \mathbb{R}$, with $\text{Im } \sqrt{z} > 0$, we denote by $\hat{\Psi}_\varepsilon$ the vector $\hat{\Psi}_\varepsilon \equiv (\hat{\psi}_{1,\varepsilon}, \hat{\psi}_{2,\varepsilon}, \hat{\psi}_{v,\varepsilon})$, where the functions $\hat{\psi}_{1,\varepsilon}$, $\hat{\psi}_{2,\varepsilon}$ and $\hat{\psi}_{v,\varepsilon}$ are defined by

$$(3.9) \quad \hat{\psi}_{1,\varepsilon}(s, u) := [(r_0(z)f_1)(s) + q_{1,\varepsilon}e^{i\sqrt{z}s}] \chi_n(u)$$

$$(3.10) \quad \hat{\psi}_{2,\varepsilon}(s, u) := [(r_0(z)f_2)(s) + q_{2,\varepsilon}e^{i\sqrt{z}s}] \chi_n(u)$$

$$(3.11) \quad \hat{\psi}_{v,\varepsilon}(s, u) := \varepsilon [\xi_{1,\varepsilon} r_v(\varepsilon^2 z; s, -1) + \xi_{2,\varepsilon} r_v(\varepsilon^2 z; s, 1)] \chi_n(u),$$

and where we set

$$(3.12) \quad (r_0(z)f_j)(s) := \int_0^\infty \left(\frac{e^{i\sqrt{z}|s-s'|}}{2i\sqrt{z}} - \frac{e^{i\sqrt{z}s}e^{i\sqrt{z}s'}}{2i\sqrt{z}} \right) f_j(s') ds'; \quad \text{Im } \sqrt{z} > 0, \quad j = 1, 2.$$

The constants $q_{1,\varepsilon}$, $q_{2,\varepsilon}$, $\xi_{1,\varepsilon}$ and $\xi_{2,\varepsilon}$ are fixed by the relations

$$(3.13) \quad \begin{aligned} \xi_{1,\varepsilon} &:= (p_1 + i\sqrt{z}q_{1,\varepsilon}); & p_1 &:= (r_0(z)f_1)'(0) \\ \xi_{2,\varepsilon} &:= (p_2 + i\sqrt{z}q_{2,\varepsilon}); & p_2 &:= (r_0(z)f_2)'(0) \end{aligned}$$

and

$$(3.14) \quad \begin{pmatrix} q_{1,\varepsilon} \\ q_{2,\varepsilon} \end{pmatrix} := \begin{pmatrix} \varepsilon r_v(\varepsilon^2 z; -1, -1) & \varepsilon r_v(\varepsilon^2 z; -1, 1) \\ \varepsilon r_v(\varepsilon^2 z; 1, -1) & \varepsilon r_v(\varepsilon^2 z; 1, 1) \end{pmatrix} \begin{pmatrix} \xi_{1,\varepsilon} \\ \xi_{2,\varepsilon} \end{pmatrix}$$

Our main result is expressed in the following Theorem 1 and Theorem 2, the proofs of which are postponed to Section 5 and Section 4 respectively.

Theorem 1. *For any vector $\tilde{\Xi}_n \equiv (f_1 \chi_n, f_2 \chi_n, 0)$ with $f_1, f_2 \in L^2((0, \infty))$ let us take $\hat{\Psi}_\varepsilon$ as it was done in Definition 3.1. Then $\hat{\Psi}_\varepsilon \in D(\tilde{H}_{\delta,\varepsilon})$, moreover for all $z \in \mathbb{C} \setminus \mathbb{R}$ there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and for all $0 < \delta \leq \varepsilon$ the following estimates hold true:*

Case 1.

$$\left\| \left[\tilde{H}_{\delta,\varepsilon} - \frac{n^2 \pi^2}{\delta^2} - z \right] \hat{\Psi}_\varepsilon - \tilde{\Xi}_n \right\|_{\tilde{\mathbb{H}}_\varepsilon} \leq c \frac{\delta}{\varepsilon^{3/2}} \|\tilde{\Xi}_n\|_{\tilde{\mathbb{H}}_\varepsilon};$$

Case 2.

$$\left\| \left[\tilde{H}_{\delta,\varepsilon} - \frac{n^2 \pi^2}{\delta^2} - z \right] \hat{\Psi}_\varepsilon - \tilde{\Xi}_n \right\|_{\tilde{\mathbb{H}}_\varepsilon} \leq c \left(\frac{\delta}{\varepsilon^{3/2}} + \frac{\delta}{\varepsilon^{5/2}} \right) \|\tilde{\Xi}_n\|_{\tilde{\mathbb{H}}_\varepsilon};$$

where c is a constant which does not depend on ε , f_1 , f_2 and n .

Remark 3.2. We notice the following estimate for the resolvent of $\tilde{H}_{\delta,\varepsilon}$: if $\tilde{\Xi}_n \equiv (f_1 \chi_n, f_2 \chi_n, 0) \in \tilde{\mathbb{H}}_\varepsilon$, then

$$\left| \left(\tilde{\Xi}_n, \hat{\Psi}_\varepsilon - \left[\tilde{H}_{\delta,\varepsilon} - \frac{n^2 \pi^2}{\delta^2} - z \right]^{-1} \tilde{\Xi}_n \right)_{\tilde{\mathbb{H}}_\varepsilon} \right| \leq \frac{1}{|\operatorname{Im} z|} \|\tilde{\Xi}_n\|_{\tilde{\mathbb{H}}_\varepsilon} \left\| \left[\tilde{H}_{\delta,\varepsilon} - \frac{n^2 \pi^2}{\delta^2} - z \right] \hat{\Psi}_\varepsilon - \tilde{\Xi}_n \right\|_{\tilde{\mathbb{H}}_\varepsilon}.$$

Theorem 2 (Asymptotic behavior of the solution in the edges). *Let us take $q_{1,\varepsilon}$, $q_{2,\varepsilon}$, $\xi_{1,\varepsilon}$ and $\xi_{2,\varepsilon}$ as it was done in equations (3.13) - (3.14). Then:*

Case 1.

$$(3.15) \quad \begin{pmatrix} q_{1,\varepsilon} \\ q_{2,\varepsilon} \end{pmatrix} = \mathcal{O}(\varepsilon) \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}; \quad \begin{pmatrix} \xi_{1,\varepsilon} \\ \xi_{2,\varepsilon} \end{pmatrix} = [1 + \mathcal{O}(\varepsilon)] \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}.$$

Case 2.

$$(3.16) \quad \begin{pmatrix} q_{1,\varepsilon} \\ q_{2,\varepsilon} \end{pmatrix} = \left[\frac{i\Lambda_0}{\sqrt{z}} + \mathcal{O}(\varepsilon) \right] \begin{pmatrix} p_1 \\ p_2 \end{pmatrix};$$

$$(3.17) \quad \begin{pmatrix} \xi_{1,\varepsilon} \\ \xi_{2,\varepsilon} \end{pmatrix} = \left[\Lambda_0^\perp + \varepsilon \frac{i\sqrt{z}}{\alpha_1^2 + \alpha_2^2} \Lambda_0 + \mathcal{O}(\varepsilon^2) \right] \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

where we denoted by Λ_0 the projector

$$(3.18) \quad \Lambda_0 = \frac{1}{\alpha_1^2 + \alpha_2^2} \begin{pmatrix} \alpha_1^2 & \alpha_1 \alpha_2 \\ \alpha_1 \alpha_2 & \alpha_2^2 \end{pmatrix} \quad \text{and} \quad \Lambda_0^\perp = 1 - \Lambda_0.$$

Here and in the following for all $a \geq 0$ we denote by $\mathcal{O}(\varepsilon^a)$ a 2×2 matrix such that $\|\mathcal{O}(\varepsilon^a)\|_{\mathcal{B}(\mathbb{C}^2)} \leq c\varepsilon^a$, for all $0 < \varepsilon < \varepsilon_0$, where c is a positive constant which does not depend on ε , p_1 and p_2 .

4. ASYMPTOTIC BEHAVIOR OF THE SOLUTION IN THE EDGES (PROOF OF THEOREM 2)

We devote this section to the proof of Theorem 2. We start with the proof of the following proposition:

Proposition 4.1. *For any $z \in \mathbb{C} \setminus \mathbb{R}$ there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the following estimates hold true:*

Case 1.

$$(4.1) \quad \sup_{s, s' \in [-1, 1]} [|r_v(\varepsilon^2 z; s, s')|] \leq c;$$

Case 2.

$$(4.2) \quad \sup_{s, s' \in [-1, 1]} \left[\left| r_v(\varepsilon^2 z; s, s') + \frac{y^*(s)y^*(s')}{\varepsilon^2 z} \right| \right] \leq c.$$

where c is a constant which does not depend on ε .

Proof. We notice that in *Case 1* the series $\sum_n \frac{1}{|\lambda_n|}$ is convergent and we use the formula (3.3) for the integral kernel $r_v(z; s, s')$. Then, in *Case 1*,

$$|r_v(\varepsilon^2 z; s, s')| = \left| \sum_n \frac{y_n(s)y_n(s')}{\lambda_n - \varepsilon^2 z} \right| \leq \sum_n \frac{|y_n(s)||y_n(s')|}{|\lambda_n - \varepsilon^2 z|}.$$

Then we use the fact that $\|y_n\|_{v, s, \infty} \leq c$ where c does not depend on n , see, e.g., [36], and the fact that for ε small enough $|\lambda_n - \varepsilon^2 z| > 2|\lambda_n|$. We obtain

$$|r_v(\varepsilon^2 z; s, s')| \leq \max_n [\|y_n\|_{v, s, \infty}^2] \sum_n \frac{1}{2|\lambda_n|} \leq c.$$

To prove the estimate (4.2) we use again the formula (3.3); we write the integral kernel $r_v(z; s, s')$ as

$$r_v(\varepsilon^2 z; s, s') = -\frac{y^*(s)y^*(s')}{\varepsilon^2 z} + \sum_{n \neq n^*} \frac{y_n(s)y_n(s')}{\lambda_n - \varepsilon^2 z}.$$

We recall that we denoted by n^* the integer associated to the eigenvalue zero $\lambda_{n^*} = 0$. The estimate (4.2) follows from the fact that the series $\sum_{n \neq n^*} \frac{1}{|\lambda_n|}$ is convergent and by the same argument used for the analysis of *Case 1*. \square

Remark 4.2. We notice that Proposition 4.1 implies that there exists ε_0 such that for all $0 < \varepsilon < \varepsilon_0$:

Case 1.

$$\|r_v(\varepsilon^2 z; \cdot, \pm 1)\|_{v, s} \leq 2 \left(\sup_{s, s' \in [-1, 1]} [|r_v(\varepsilon^2 z; s, s')|] \right) \leq c;$$

Case 2.

$$\left\| r_v(\varepsilon^2 z; \cdot, \pm 1) + \frac{y^*y^*(\pm 1)}{\varepsilon^2 z} \right\|_{v, s} \leq 2 \left(\sup_{s, s' \in [-1, 1]} \left[\left| r_v(\varepsilon^2 z; s, s') + \frac{y^*(s)y^*(s')}{\varepsilon^2 z} \right| \right] \right) \leq c.$$

where c is a constant which does not depend on ε .

We are now ready to give the proof of Theorem 2:

Proof of Theorem 2. We use the equalities (3.13) in equation (3.14) and obtain

$$\begin{pmatrix} q_{1, \varepsilon} \\ q_{2, \varepsilon} \end{pmatrix} = \begin{pmatrix} \varepsilon r_v(\varepsilon^2 z; -1, -1) & \varepsilon r_v(\varepsilon^2 z; -1, 1) \\ \varepsilon r_v(\varepsilon^2 z; 1, -1) & \varepsilon r_v(\varepsilon^2 z; 1, 1) \end{pmatrix} \begin{pmatrix} p_1 + i\sqrt{z}q_{1, \varepsilon} \\ p_2 + i\sqrt{z}q_{2, \varepsilon} \end{pmatrix}.$$

We denote by Λ_ε the matrix

$$\Lambda_\varepsilon := \begin{pmatrix} r_v(\varepsilon^2 z; -1, -1) & r_v(\varepsilon^2 z; -1, 1) \\ r_v(\varepsilon^2 z; 1, -1) & r_v(\varepsilon^2 z; 1, 1) \end{pmatrix}.$$

By Proposition 4.1, for ε small enough the matrix $(1 - i\varepsilon\sqrt{z}\Lambda_\varepsilon)^{-1}$ is well defined, and we get the formula

$$\begin{pmatrix} q_{1,\varepsilon} \\ q_{2,\varepsilon} \end{pmatrix} = (1 - i\varepsilon\sqrt{z}\Lambda_\varepsilon)^{-1} \varepsilon \Lambda_\varepsilon \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}.$$

Then in *Case 1*, the estimate (4.1) gives

$$|r_v(\varepsilon^2 z; (-1)^j, (-1)^k)| \leq c \quad j, k = 1, 2$$

which implies $\Lambda_\varepsilon = \mathcal{O}(1)$ and, consequently, the first estimate in equation (3.15). The second estimate in the same equation follows directly from the definition of $\xi_{1,\varepsilon}$ and $\xi_{2,\varepsilon}$, see formula (3.13).

In *Case 2* we use the formula

$$r_v(\varepsilon^2 z; (-1)^j, (-1)^k) = -\frac{\alpha_j \alpha_k}{\varepsilon^2 z} + \sum_{n \neq n^*} \frac{y_n((-1)^j) y_n((-1)^k)}{\lambda_n - \varepsilon^2 z} \quad j, k = 1, 2,$$

see the estimate (4.2) in Proposition 4.1. Reminding that $y^*((-1)^k) = \alpha_k$, for $k = 1, 2$, it follows that

$$\varepsilon \Lambda_\varepsilon = -\frac{1}{\varepsilon z} \begin{pmatrix} \alpha_1^2 & \alpha_1 \alpha_2 \\ \alpha_1 \alpha_2 & \alpha_2^2 \end{pmatrix} + \varepsilon \begin{pmatrix} \sum_{n \neq n^*} \frac{y_n(-1) y_n(-1)}{\lambda_n - \varepsilon^2 z} & \sum_{n \neq n^*} \frac{y_n(-1) y_n(1)}{\lambda_n - \varepsilon^2 z} \\ \sum_{n \neq n^*} \frac{y_n(1) y_n(-1)}{\lambda_n - \varepsilon^2 z} & \sum_{n \neq n^*} \frac{y_n(1) y_n(1)}{\lambda_n - \varepsilon^2 z} \end{pmatrix} = -\frac{\alpha_1^2 + \alpha_2^2}{\varepsilon z} \Lambda_0 + \varepsilon \tilde{\Lambda}_1,$$

where the matrix $\tilde{\Lambda}_1 = \mathcal{O}(1)$ (see Proposition 4.1). We get the following formula for the operator $(1 - i\varepsilon\sqrt{z}\Lambda_\varepsilon)^{-1} \varepsilon \Lambda_\varepsilon$

$$(1 - i\varepsilon\sqrt{z}\Lambda_\varepsilon)^{-1} \varepsilon \Lambda_\varepsilon = \left[i \frac{\alpha_1^2 + \alpha_2^2}{\sqrt{z}} \Lambda_0 + \varepsilon - i\varepsilon^2 \sqrt{z} \tilde{\Lambda}_1 \right]^{-1} \left[-\frac{\alpha_1^2 + \alpha_2^2}{z} \Lambda_0 + \varepsilon^2 \tilde{\Lambda}_1 \right].$$

Let A be a 2×2 matrix and a some complex constant not equal to zero, moreover assume that the matrix $(a + A)$ is invertible, then the formula $(a + A)^{-1} = 1/a - (a + A)^{-1} A a^{-1}$ holds true. We use the latter formula to obtain:

$$(1 - i\varepsilon\sqrt{z}\Lambda_\varepsilon)^{-1} \varepsilon \Lambda_\varepsilon = i \frac{\Lambda_0}{\sqrt{z}} - \varepsilon \left[i \frac{\alpha_1^2 + \alpha_2^2}{\sqrt{z}} \Lambda_0 + \varepsilon - i\varepsilon^2 \sqrt{z} \tilde{\Lambda}_1 \right]^{-1} i \frac{\Lambda_0}{\sqrt{z}} + \varepsilon^2 \tilde{\Lambda}_2.$$

Here we used $\Lambda_0^\perp \Lambda_0 = 0$ and set

$$\begin{aligned} \varepsilon^2 \tilde{\Lambda}_2 &= \left[i \frac{\alpha_1^2 + \alpha_2^2}{\sqrt{z}} \right]^{-1} [\varepsilon^2 \tilde{\Lambda}_1] + \left[i \frac{\alpha_1^2 + \alpha_2^2}{\sqrt{z}} \Lambda_0 + \varepsilon - i\varepsilon^2 \sqrt{z} \tilde{\Lambda}_1 \right]^{-1} \Lambda_0^\perp [\varepsilon^2 \tilde{\Lambda}_1] \\ &\quad - \left[i \frac{\alpha_1^2 + \alpha_2^2}{\sqrt{z}} \Lambda_0 + \varepsilon - i\varepsilon^2 \sqrt{z} \tilde{\Lambda}_1 \right]^{-1} [-i\varepsilon^2 \sqrt{z} \tilde{\Lambda}_1] \left[i \frac{\alpha_1^2 + \alpha_2^2}{\sqrt{z}} \right]^{-1} \left[-\frac{\alpha_1^2 + \alpha_2^2}{z} \Lambda_0 + \varepsilon^2 \tilde{\Lambda}_1 \right] \\ &\quad - \varepsilon \left[i \frac{\alpha_1^2 + \alpha_2^2}{\sqrt{z}} \Lambda_0 + \varepsilon - i\varepsilon^2 \sqrt{z} \tilde{\Lambda}_1 \right]^{-1} \left[i \frac{\alpha_1^2 + \alpha_2^2}{\sqrt{z}} \right]^{-1} [\varepsilon^2 \tilde{\Lambda}_1]. \end{aligned}$$

We remark that for ε small enough one has $\|\tilde{\Lambda}_2\|_{\mathcal{B}(\mathbb{C}^2)} \leq c$, where the constant c does not depend on ε . By using again the formula $(a + A)^{-1} = 1/a - (a + A)^{-1} A a^{-1}$ we get

$$\begin{aligned} & -\varepsilon \left[i \frac{\alpha_1^2 + \alpha_2^2}{\sqrt{z}} \Lambda_0 + \varepsilon - i\varepsilon^2 \sqrt{z} \tilde{\Lambda}_1 \right]^{-1} i \frac{\Lambda_0}{\sqrt{z}} \\ &= -\varepsilon \frac{\Lambda_0}{\alpha_1^2 + \alpha_2^2} + \varepsilon \left[i \frac{\alpha_1^2 + \alpha_2^2}{\sqrt{z}} \Lambda_0 + \varepsilon - i\varepsilon^2 \sqrt{z} \tilde{\Lambda}_1 \right]^{-1} \left[\varepsilon - i\varepsilon^2 \sqrt{z} \tilde{\Lambda}_1 \right] \frac{\Lambda_0}{\alpha_1^2 + \alpha_2^2}. \end{aligned}$$

Hence we have

$$(1 - i\varepsilon\sqrt{z}\Lambda_\varepsilon)^{-1}\varepsilon\Lambda_\varepsilon = i\frac{\Lambda_0}{\sqrt{z}} - \varepsilon\frac{\Lambda_0}{\alpha_1^2 + \alpha_2^2} + \varepsilon^2\tilde{\Lambda}_3,$$

where $\tilde{\Lambda}_3$ is given by

$$\varepsilon^2\tilde{\Lambda}_3 = \varepsilon^2\tilde{\Lambda}_2 + \varepsilon\left[i\frac{\alpha_1^2 + \alpha_2^2}{\sqrt{z}}\Lambda_0 + \varepsilon - i\varepsilon^2\sqrt{z}\tilde{\Lambda}_1\right]^{-1}\left[\varepsilon - i\varepsilon^2\sqrt{z}\tilde{\Lambda}_1\right]\frac{\Lambda_0}{\alpha_1^2 + \alpha_2^2}.$$

We notice that for ε small enough one has $\|\tilde{\Lambda}_3\|_{\mathcal{B}(\mathbb{C}^2)} \leq c$, where the constant c does not depend on ε . Hence we obtain, for ε small enough, the following formula for $q_{1,\varepsilon}$ and $q_{2,\varepsilon}$

$$(4.3) \quad \begin{pmatrix} q_{1,\varepsilon} \\ q_{2,\varepsilon} \end{pmatrix} = \left[i\frac{\Lambda_0}{\sqrt{z}} - \varepsilon\frac{\Lambda_0}{\alpha_1^2 + \alpha_2^2} + \varepsilon^2\tilde{\Lambda}_3 \right] \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

which yields the estimate (3.16). The estimate (3.17) comes from the definition of $\xi_{1,\varepsilon}$ and $\xi_{2,\varepsilon}$, see equation (3.13), and from equation (4.3). \square

5. ASYMPTOTIC BEHAVIOR OF THE SOLUTION IN THE VERTEX REGION (PROOF OF THEOREM 1)

We devote this section to the proof of Theorem 1. We first give some preliminary estimates on the function $\hat{\psi}_{v,\varepsilon}$, see Proposition 5.3 below.

For any $z \in \mathbb{C} \setminus \mathbb{R}$, we denote by $r_v^{(0)}(z)$ the resolvent of the Neumann Laplacian in $L^2((-1, 1))$. The integral kernel of $r_v^{(0)}(z)$ can be derived from formulas (3.4) - (3.5) by setting $\gamma = 0$; a straightforward calculation gives

$$(5.1) \quad r_v^{(0)}(z; s, s') = \begin{cases} -\frac{\cos(\sqrt{z}(s+1))\cos(\sqrt{z}(s'-1))}{\sqrt{z}\sin(2\sqrt{z})} & s \leq s' \\ -\frac{\cos(\sqrt{z}(s-1))\cos(\sqrt{z}(s'+1))}{\sqrt{z}\sin(2\sqrt{z})} & s > s' \end{cases}$$

Proposition 5.1. *For any $z \in \mathbb{C} \setminus \mathbb{R}$ there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the following estimates hold true:*

$$(5.2) \quad \left\| \frac{d}{ds} r_v^{(0)}(\varepsilon^2 z; \cdot, -1) \right\|_{v,s} \leq c; \quad \left\| \frac{d}{ds} r_v^{(0)}(\varepsilon^2 z; \cdot, 1) \right\|_{v,s} \leq c;$$

where c is a constant which does not depend on ε . Moreover for any $g \in L^2((-1, 1))$

$$(5.3) \quad \left\| \frac{d}{ds} r_v^{(0)}(\varepsilon^2 z)g \right\|_{v,s} \leq c\|g\|_{v,s},$$

where c is a constant which does not depend on ε .

Proof. By a direct computation

$$(5.4) \quad \left\| \frac{d}{ds} r_v^{(0)}(\varepsilon^2 z; \cdot, \pm 1) \right\|_{v,s}^2 = \int_{-1}^1 \left| \frac{d}{ds} \frac{\cos(\sqrt{\varepsilon^2 z}(s \pm 1))}{\sqrt{\varepsilon^2 z} \sin(2\sqrt{\varepsilon^2 z})} \right|^2 ds = \frac{\|\sin(\sqrt{\varepsilon^2 z}(\cdot \pm 1))\|_{v,s}^2}{|\sin(2\sqrt{\varepsilon^2 z})|^2}$$

For ε small enough, the norms $\|\sin(\sqrt{\varepsilon^2 z}(\cdot - 1))\|_{v,s}$ and $\|\sin(\sqrt{\varepsilon^2 z}(\cdot + 1))\|_{v,s}$ can be bounded by

$$(5.5) \quad \|\sin(\sqrt{\varepsilon^2 z}(\cdot - 1))\|_{v,s}^2 = \int_{-1}^1 \left| \frac{e^{i\sqrt{\varepsilon^2 z}(s-1)} - e^{-i\sqrt{\varepsilon^2 z}(s-1)}}{2} \right|^2 ds \leq c\varepsilon^2.$$

and similarly

$$(5.6) \quad \|\sin(\sqrt{\varepsilon^2 z}(\cdot + 1))\|_{v,s}^2 \leq c\varepsilon^2,$$

where c is a constant which does not depend on ε . Using the latter estimates in equations (5.4) we get

$$\left\| \frac{d}{ds} r_v^{(0)}(\varepsilon^2 z; \cdot, \pm 1) \right\|_{v,s}^2 \leq \frac{c\varepsilon^2}{|\sin(2\sqrt{\varepsilon^2 z})|^2},$$

which imply estimates (5.2). Also the estimate (5.3) can be obtained through a direct calculation. For any $g \in L^2((-1, 1), ds)$

$$\begin{aligned} \frac{d}{ds} (r_v^{(0)}(z)g)(s) &= \frac{\sin(\sqrt{z}(s+1))}{\sin(2\sqrt{z})} \int_s^1 \cos(\sqrt{z}(s'-1))g(s')ds' \\ &\quad + \frac{\sin(\sqrt{z}(s-1))}{\sin(2\sqrt{z})} \int_{-1}^s \cos(\sqrt{z}(s'+1))g(s')ds'. \end{aligned}$$

Then

$$\begin{aligned} (5.7) \quad \left\| \frac{d}{ds} r_v^{(0)}(\varepsilon^2 z)g \right\|_{v,s} &\leq \left\| \frac{\sin(\sqrt{\varepsilon^2 z}(\cdot+1))}{\sin(2\sqrt{\varepsilon^2 z})} \int_{\cdot}^1 \cos(\sqrt{\varepsilon^2 z}(s'-1))g(s')ds' \right\|_{v,s} \\ &\quad + \left\| \frac{\sin(\sqrt{\varepsilon^2 z}(\cdot-1))}{\sin(2\sqrt{\varepsilon^2 z})} \int_{-1}^{\cdot} \cos(\sqrt{\varepsilon^2 z}(s'+1))g(s')ds' \right\|_{v,s}. \end{aligned}$$

From the Cauchy-Schwarz inequality, for all $s \in (-1, 1)$,

$$\left| \int_s^1 \cos(\sqrt{\varepsilon^2 z}(s' \pm 1))g(s')ds' \right| \leq c\|g\|_{v,s},$$

which implies

$$\left\| \frac{\sin(\sqrt{\varepsilon^2 z}(\cdot \pm 1))}{\sin(2\sqrt{\varepsilon^2 z})} \int_{\cdot}^1 \cos(\sqrt{\varepsilon^2 z}(s' \mp 1))g(s')ds' \right\|_{v,s} \leq c \frac{1}{|\sin(2\sqrt{\varepsilon^2 z})|} \|\sin(\sqrt{\varepsilon^2 z}(\cdot \pm 1))\|_{v,s} \|g\|_{v,s}.$$

Using again the estimates (5.5) and (5.6) we obtain

$$\left\| \frac{\sin(\sqrt{\varepsilon^2 z}(\cdot \pm 1))}{\sin(2\sqrt{\varepsilon^2 z})} \int_{\cdot}^1 \cos(\sqrt{\varepsilon^2 z}(s' \mp 1))g(s')ds' \right\|_{v,s} \leq c\|g\|_{v,s},$$

and the estimate (5.3) follows from the last estimate and from equation (5.7). \square

Proposition 5.2. *For any $z \in \mathbb{C} \setminus \mathbb{R}$ there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the following estimates hold true:*

Case 1.

$$(5.8) \quad \left\| \frac{d}{ds} r_v(\varepsilon^2 z; \cdot, -1) \right\|_{v,s} \leq c, \quad \left\| \frac{d}{ds} r_v(\varepsilon^2 z; \cdot, 1) \right\|_{v,s} \leq c;$$

Case 2.

$$(5.9) \quad \left\| \frac{d}{ds} r_v(\varepsilon^2 z; \cdot, -1) + \frac{y^{*'} y^*(-1)}{\varepsilon^2 z} \right\|_{v,s} \leq c, \quad \left\| \frac{d}{ds} r_v(\varepsilon^2 z; \cdot, 1) + \frac{y^{*'} y^*(-1)}{\varepsilon^2 z} \right\|_{v,s} \leq c.$$

where c is a constant which does not depend on ε .

Proof. To prove the estimate (5.8) we use the well known resolvent identity

$$\begin{aligned} (5.10) \quad r_v(\varepsilon^2 z; s, -1) &= [r_v(\varepsilon^2 z)\delta(\cdot+1)](s) = [(r_v^{(0)}(\varepsilon^2 z) - r_v^{(0)}(\varepsilon^2 z)(-\gamma^2/4)r_v(\varepsilon^2 z))\delta(\cdot+1)] \\ &= r_v^{(0)}(\varepsilon^2 z; s, -1) - [r_v^{(0)}(\varepsilon^2 z)(-\gamma^2/4)r_v(\varepsilon^2 z; \cdot, -1)](s). \end{aligned}$$

Taking the derivative of the latter expression we obtain

$$\begin{aligned} \left\| \frac{d}{ds} r_v(\varepsilon^2 z; \cdot, -1) \right\|_{v,s} &= \left\| \frac{d}{ds} r_v^{(0)}(\varepsilon^2 z; \cdot, -1) - \frac{d}{ds} [r_v^{(0)}(\varepsilon^2 z)(-\gamma^2/4)r_v(\varepsilon^2 z; \cdot, -1)] \right\|_{v,s} \\ &\leq c[1 + \|r_v(\varepsilon^2 z; \cdot, -1)\|_{v,s}], \end{aligned}$$

where we used the triangle inequality, estimates (5.2) and (5.3), and the fact that γ is bounded. As we have proven that in *Case 1*, $\|r_v(\varepsilon^2 z; \cdot, -1)\|_{v,s} \leq c$, see Remark 4.2, we get the first estimate in the equation (5.8), the proof of the second one is similar and we omit it.

To prove the estimates (5.9), we rewrite the equation (5.10) as

$$(5.11) \quad \begin{aligned} r_v(\varepsilon^2 z; s, -1) = & r_v^{(0)}(\varepsilon^2 z; s, -1) - \left[r_v^{(0)}(\varepsilon^2 z)(-\gamma^2/4) \left(r_v(\varepsilon^2 z; \cdot, -1) + \frac{y^* y^*(-1)}{\varepsilon^2 z} \right) \right](s) \\ & + \left[r_v^{(0)}(\varepsilon^2 z)(-\gamma^2/4) \frac{y^* y^*(-1)}{\varepsilon^2 z} \right](s). \end{aligned}$$

Then we use the fact that y^* is the solution of the problem (3.6) and we get

$$\left[r_v^{(0)}(\varepsilon^2 z)(-\gamma^2/4) \frac{y^* y^*(-1)}{\varepsilon^2 z} \right](s) = \left[r_v^{(0)}(\varepsilon^2 z) \frac{d^2}{ds^2} \frac{y^* y^*(-1)}{\varepsilon^2 z} \right](s).$$

But

$$\begin{aligned} \left[r_v^{(0)}(\varepsilon^2 z) \frac{d^2}{ds^2} y^* \right](s) &= \int_{-1}^1 r_v^{(0)}(\varepsilon^2 z; s, s') \frac{d^2}{ds'^2} y^*(s') ds' = - \int_{-1}^1 \frac{d}{ds'} r_v^{(0)}(\varepsilon^2 z; s, s') \frac{d}{ds'} y^*(s') ds' \\ &= \int_{-1}^1 \frac{d^2}{ds'^2} r_v^{(0)}(\varepsilon^2 z; s, s') y^*(s') ds' = -\varepsilon^2 z [r_v^{(0)}(\varepsilon^2 z) y^*](s) - y^*(s), \end{aligned}$$

where we used the fact that the boundary terms are null because

$$\left. \frac{d}{ds} y^* \right|_{s=\pm 1} = 0; \quad \left. \frac{d}{ds'} r_v^{(0)}(\varepsilon^2 z; s, \cdot) \right|_{s'=\pm 1} = 0;$$

and we also used the fact that

$$\frac{d^2}{ds'^2} r_v^{(0)}(\varepsilon^2 z; s, s') = -\varepsilon^2 z r_v^{(0)}(\varepsilon^2 z; s, s') - \delta(s - s').$$

We remark that $r_v^{(0)}(z; s, s') = r_v^{(0)}(z; s', s)$, see equation (5.1). We can rewrite the equation (5.11) as

$$\begin{aligned} r_v(\varepsilon^2 z; s, -1) = & r_v^{(0)}(\varepsilon^2 z; s, -1) - \left[r_v^{(0)}(\varepsilon^2 z)(-\gamma^2/4) \left(r_v(\varepsilon^2 z; \cdot, -1) + \frac{y^* y^*(-1)}{\varepsilon^2 z} \right) \right](s) \\ & - (\varepsilon^2 z [r_v^{(0)}(\varepsilon^2 z) y^*](s) + y^*(s)) \frac{y^*(-1)}{\varepsilon^2 z} \end{aligned}$$

or equivalently

$$\begin{aligned} & r_v(\varepsilon^2 z; s, -1) + y^*(s) \frac{y^*(-1)}{\varepsilon^2 z} \\ &= r_v^{(0)}(\varepsilon^2 z; s, -1) - \left[r_v^{(0)}(\varepsilon^2 z)(-\gamma^2/4) \left(r_v(\varepsilon^2 z; \cdot, -1) + \frac{y^* y^*(-1)}{\varepsilon^2 z} \right) \right](s) \\ & \quad - [r_v^{(0)}(\varepsilon^2 z) y^* y^*(-1)](s). \end{aligned}$$

Now we can give an estimate of the derivative of the latter equation

$$\begin{aligned} & \left\| \frac{d}{ds} \left[r_v(\varepsilon^2 z; \cdot, -1) \right] + \frac{y^{*'} y^*(-1)}{\varepsilon^2 z} \right\|_{v,s} \\ & \leq \left\| \frac{d}{ds} r_v^{(0)}(\varepsilon^2 z; \cdot, -1) \right\|_{v,s} + \left\| \frac{d}{ds} \left[r_v^{(0)}(\varepsilon^2 z)(-\gamma^2/4) \left(r_v(\varepsilon^2 z; \cdot, -1) + \frac{y^* y^*(-1)}{\varepsilon^2 z} \right) \right] \right\|_{v,s} \\ & \quad + \left\| \frac{d}{ds} \left[r_v^{(0)}(\varepsilon^2 z) y^* y^*(-1) \right] \right\|_{v,s} \leq c, \end{aligned}$$

where we used the triangle inequality, estimates (5.2) and (5.3), and the fact that γ is bounded. Moreover we also used the result stated in Remark 4.2 and the fact that $\|y^*\|_{v,s} = 1$. The proof of the second estimate in equation (5.9) is identical and we omit it. \square

Proposition 5.3. *Let $\hat{\psi}_{v,\varepsilon}$ be as in equation (3.11). Then there exist $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the following estimates hold true:*

Case 1

$$(5.12) \quad \|\hat{\psi}_{v,\varepsilon}\|_{L^2(V)} \leq c\varepsilon(|\xi_{1,\varepsilon}| + |\xi_{2,\varepsilon}|), \quad \left\| \frac{\partial}{\partial s} \hat{\psi}_{v,\varepsilon} \right\|_{L^2(V)} \leq c\varepsilon(|\xi_{1,\varepsilon}| + |\xi_{2,\varepsilon}|);$$

Case 2

$$(5.13) \quad \|\hat{\psi}_{v,\varepsilon} - \hat{\psi}_\varepsilon^*\|_{L^2(V)} \leq c\varepsilon(|\xi_{1,\varepsilon}| + |\xi_{2,\varepsilon}|), \quad \left\| \frac{\partial}{\partial s} [\hat{\psi}_{v,\varepsilon} - \hat{\psi}_\varepsilon^*] \right\|_{L^2(V)} \leq c\varepsilon(|\xi_{1,\varepsilon}| + |\xi_{2,\varepsilon}|),$$

with

$$\hat{\psi}_\varepsilon^*(s, u) = -\frac{1}{\varepsilon} \frac{y^*(s)}{z} [\xi_{1,\varepsilon} y^*(-1) + \xi_{2,\varepsilon} y^*(1)] \chi_n(u).$$

where c is a constant which does not depend on ε , f_1 , f_2 and n .

Proof. The proposition is a direct consequence of the definition of $\hat{\psi}_{v,\varepsilon}$, see equation (3.11), and of Remark 4.2 and Proposition 5.2. \square

The following remark makes sense only in the *Case 2*:

Remark 5.4. From Theorem 2, equation (3.17), and since $y^*(-1) = \alpha_1$ and $y^*(1) = \alpha_2$ we have that there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$

$$\frac{1}{\varepsilon} |\xi_{1,\varepsilon} y^*(-1) + \xi_{2,\varepsilon} y^*(1)| \leq c(|p_1| + |p_2|)$$

where c is a constant which does not depend on ε , p_1 and p_2 . Thus implying

$$(5.14) \quad \|\hat{\psi}_\varepsilon^*\|_{L^2(V)} \leq c(|p_1| + |p_2|); \quad \left\| \frac{\partial}{\partial s} \hat{\psi}_\varepsilon^* \right\|_{L^2(V)} \leq c(|p_1| + |p_2|).$$

The following two propositions make preparations for the proof of Theorem 1.

Proposition 5.5 (Regularity properties of $\hat{\Psi}_\varepsilon$). *For any vector $\tilde{\Xi}_n \equiv (f_1 \chi_n, f_2 \chi_n, 0)$ with $f_1, f_2 \in L^2((0, \infty))$ let us take $\hat{\Psi}_\varepsilon$ as it was done in Definition 3.1. Then $\hat{\Psi}_\varepsilon \in D(\tilde{H}_{\delta,\varepsilon})$.*

Proof. From the definition of $r_v(z)$, see equation (3.5), it follows that the functions $r_v(\varepsilon^2 z; s, \pm 1)$ are well defined and that, for any $\varepsilon > 0$, $r_v(\varepsilon^2 z; \cdot, \pm 1) \in H^2((-1, 1))$. Then for any $0 < \delta \leq \varepsilon \leq 1$ and from the definition of $\hat{\psi}_{1,\varepsilon}$, $\hat{\psi}_{2,\varepsilon}$ and $\hat{\psi}_{v,\varepsilon}$, see equations (3.9) - (3.11), it follows that, $\|\hat{\Psi}\|_{\tilde{H}_{\delta,\varepsilon}^2} < \infty$.

It remains to prove that the functions $\hat{\psi}_{1,\varepsilon}$, $\hat{\psi}_{2,\varepsilon}$ and $\hat{\psi}_{v,\varepsilon}$ satisfy the boundary conditions given in the definition of $D(\tilde{H}_{\delta,\varepsilon})$, see equation (2.12). The boundary conditions in $u = 0, 1$ are trivially satisfied because of the definition of the function $\chi_n(u)$, see equation (3.8).

Since the constants $q_{1,\varepsilon}$ and $q_{2,\varepsilon}$ are related to the constants $\xi_{1,\varepsilon}$ and $\xi_{2,\varepsilon}$ through the relation (3.14), one has that the functions $\hat{\psi}_{1,\varepsilon}$, $\hat{\psi}_{2,\varepsilon}$ and $\hat{\psi}_{v,\varepsilon}$ satisfy the conditions:

$$(5.15) \quad \hat{\psi}_{v,\varepsilon}(-1, u) = q_{1,\varepsilon} \chi_n(u) = \hat{\psi}_{1,\varepsilon}(0, u)$$

$$(5.16) \quad \hat{\psi}_{v,\varepsilon}(1, u) = q_{2,\varepsilon} \chi_n(u) = \hat{\psi}_{2,\varepsilon}(0, u).$$

From the definition of $r_v(z)$, see equation (3.5), one has

$$(5.17) \quad \frac{d}{ds} r_v(z; s, 1) = \frac{d}{ds} \frac{\zeta_v(z; s) \eta_v(z; 1)}{\mathcal{W}_v(z; 1)} = \frac{\zeta'_v(z; s)}{\zeta'_v(z; 1)}$$

$$(5.18) \quad \frac{d}{ds} r_v(z; s, -1) = \frac{d}{ds} \frac{\eta_v(z; s) \zeta_v(z; -1)}{\mathcal{W}_v(z; -1)} = -\frac{\eta'_v(z; s)}{\eta'_v(z; -1)},$$

where we used $\mathcal{W}_v(z) = \mathcal{W}_v(z; 1) = \mathcal{W}_v(z; -1)$ in equations (5.17) and (5.18) respectively. Since $\zeta'_v(z; -1) = 0$ and $\eta'_v(z; 1) = 0$ the following relations hold true

$$\begin{aligned} \left. \frac{d}{ds} r_v(z; s, 1) \right|_{s=1} &= 1; & \left. \frac{d}{ds} r_v(z; s, -1) \right|_{s=-1} &= -1; \\ \left. \frac{d}{ds} r_v(z; s, 1) \right|_{s=-1} &= \left. \frac{d}{ds} r_v(z; s, -1) \right|_{s=1} = 0. \end{aligned}$$

Hence

$$(5.19) \quad \frac{\partial}{\partial s} \hat{\psi}_{1,\varepsilon}(0, u) = (r_0(z)f_1)'(0)\chi_n(u) + i\sqrt{z}q_{1,\varepsilon}\chi_n(u) = \xi_{1,\varepsilon}\chi_n(u) = -\frac{1}{\varepsilon} \frac{\partial}{\partial s} \hat{\psi}_{v,\varepsilon}(-1, u)$$

$$(5.20) \quad \frac{\partial}{\partial s} \hat{\psi}_{2,\varepsilon}(0, u) = (r_0(z)f_2)'(0)\chi_n(u) + i\sqrt{z}q_{2,\varepsilon}\chi_n(u) = \xi_{2,\varepsilon}\chi_n(u) = \frac{1}{\varepsilon} \frac{\partial}{\partial s} \hat{\psi}_{v,\varepsilon}(1, u),$$

and we have proven that $\hat{\Psi}_\varepsilon \in D(\tilde{H}_{\delta,\varepsilon})$. \square

Proposition 5.6. *For any couple of constants $\xi_{1,\varepsilon}$ and $\xi_{2,\varepsilon}$ let us take $\hat{\psi}_{v,\varepsilon}$ as we did in equation (3.11), then*

Case 1.

$$(5.21) \quad \left\| \left[\tilde{L}_{\delta/\varepsilon} - \frac{n^2\pi^2}{(\delta/\varepsilon)^2} - \varepsilon^2 z \right] \hat{\psi}_{v,\varepsilon} \right\|_{L^2(V)} \leq c \frac{\delta}{\varepsilon} [|\xi_{1,\varepsilon}| + |\xi_{2,\varepsilon}|] \varepsilon;$$

Case 2.

$$(5.22) \quad \left\| \left[\tilde{L}_{\delta/\varepsilon} - \frac{n^2\pi^2}{(\delta/\varepsilon)^2} - \varepsilon^2 z \right] \hat{\psi}_{v,\varepsilon} \right\|_{L^2(V)} \leq c \frac{\delta}{\varepsilon} [|\xi_{1,\varepsilon}| + |\xi_{2,\varepsilon}|] \varepsilon + \tilde{c} \frac{\delta}{\varepsilon} \left[\|\hat{\psi}_\varepsilon^*\|_{L^2(V)} + \left\| \frac{\partial}{\partial s} \hat{\psi}_\varepsilon^* \right\|_{L^2(V)} \right];$$

where $\tilde{L}_{\delta/\varepsilon}$ was defined in (2.14) and c and \tilde{c} are two constants which do not depend on ε , f_1 , f_2 and n .

Proof. We first notice that, because of the regularity properties of the function $\hat{\psi}_{v,\varepsilon}$, see Proposition 5.5, for any $0 < \delta \leq \varepsilon \leq 1$ the function $\tilde{L}_{\delta/\varepsilon} \hat{\psi}_{v,\varepsilon}$ belongs to $L^2(V)$. Then we notice that by our assumption $\gamma \in C_0^\infty((-1, 1))$ and from the definition of $g_{\delta/\varepsilon}$ and $W_{\delta/\varepsilon}$, see equations (2.1) and (2.15) respectively, one has the following estimates

$$(5.23) \quad \left\| \frac{1}{g_{\delta/\varepsilon}} - 1 \right\|_{L^\infty(V)} \leq c \frac{\delta}{\varepsilon}; \quad \left\| W_{\delta/\varepsilon} + \frac{\gamma^2}{4} \right\|_{L^\infty(V)} \leq c \frac{\delta}{\varepsilon}; \quad \left\| \frac{\partial}{\partial s} \frac{1}{g_{\delta/\varepsilon}} \right\|_{L^\infty(V)} \leq c \frac{\delta}{\varepsilon}.$$

From the triangle inequality it follows that

$$\begin{aligned} & \left\| \left[\tilde{L}_{\delta/\varepsilon} - \frac{n^2\pi^2}{(\delta/\varepsilon)^2} - \varepsilon^2 z \right] \hat{\psi}_{v,\varepsilon} \right\|_{L^2(V)} \\ & \leq \left\| \left[-\frac{1}{g_{\delta/\varepsilon}} \frac{\partial^2}{\partial s^2} - \frac{\gamma^2}{4} - \varepsilon^2 z \right] \hat{\psi}_{v,\varepsilon} \right\|_{L^2(V)} \\ & \quad + \left\| W_{\delta/\varepsilon} + \frac{\gamma^2}{4} \right\|_{L^\infty(V)} \|\hat{\psi}_{v,\varepsilon}\|_{L^2(V)} + \left\| \frac{\partial}{\partial s} \frac{1}{g_{\delta/\varepsilon}} \right\|_{L^\infty(V)} \left\| \frac{\partial}{\partial s} \hat{\psi}_{v,\varepsilon} \right\|_{L^2(V)}, \end{aligned}$$

where we used the fact that

$$-\frac{\partial^2}{\partial u^2} \hat{\psi}_{v,\varepsilon} = n^2 \pi^2 \hat{\psi}_{v,\varepsilon}.$$

By noticing that

$$\left[-\frac{1}{g_{\delta/\varepsilon}} \frac{\partial^2}{\partial s^2} - \frac{\gamma^2}{4} - \varepsilon^2 z \right] \hat{\psi}_{v,\varepsilon} = \left[\frac{1}{g_{\delta/\varepsilon}} - 1 \right] \left[\frac{\gamma^2}{4} + \varepsilon^2 z \right] \hat{\psi}_{v,\varepsilon},$$

and using the estimates in equation (5.23) we get

$$\left\| \left[\tilde{L}_{\delta/\varepsilon} - \frac{n^2\pi^2}{(\delta/\varepsilon)^2} - \varepsilon^2 z \right] \hat{\psi}_{v,\varepsilon} \right\|_{L^2(V)} \leq c \frac{\delta}{\varepsilon} \left[\|\hat{\psi}_{v,\varepsilon}\|_{L^2(V)} + \left\| \frac{\partial}{\partial s} \hat{\psi}_{v,\varepsilon} \right\|_{L^2(V)} \right].$$

In *Case 1* the statement follows from Proposition 5.3. In *Case 2* one has to use the trivial inequalities

$$\begin{aligned} \|\hat{\psi}_{v,\varepsilon}\|_{L^2(V)} &\leq \|\hat{\psi}_{v,\varepsilon} - \hat{\psi}_\varepsilon^*\|_{L^2(V)} + \|\hat{\psi}_\varepsilon^*\|_{L^2(V)} ; \\ \left\| \frac{\partial}{\partial s} \hat{\psi}_{v,\varepsilon} \right\|_{L^2(V)} &\leq \left\| \frac{\partial}{\partial s} [\hat{\psi}_{v,\varepsilon} - \hat{\psi}_\varepsilon^*] \right\|_{L^2(V)} + \left\| \frac{\partial}{\partial s} \hat{\psi}_\varepsilon^* \right\|_{L^2(V)} \end{aligned}$$

and the statement (5.22) follows from the estimates (5.13) and from Remark 5.4. \square

We are now ready to give the proof of Theorem 1.

Proof of Theorem 1. The fact that $\hat{\Psi}_\varepsilon \in D(\tilde{H}_{\delta,\varepsilon})$ was proved in Proposition 5.5. From the definition of the Hamiltonian $\tilde{H}_{\delta,\varepsilon}$ and of the vector $\hat{\Psi}_\varepsilon$ we have that

$$\left[\tilde{H}_{\delta,\varepsilon} - \frac{n^2\pi^2}{\delta^2} - z \right] \hat{\Psi}_\varepsilon = \left(f_1\chi_n, f_2\chi_n, \frac{1}{\varepsilon^2} \left[\tilde{L}_{\delta/\varepsilon} - \frac{n^2\pi^2}{(\delta/\varepsilon)^2} - \varepsilon^2 z \right] \hat{\psi}_{v,\varepsilon} \right),$$

where we used

$$\left[-\frac{\partial^2}{\partial s^2} - \frac{1}{\delta^2} \frac{\partial^2}{\partial u^2} - \frac{n^2\pi^2}{\delta^2} - z \right] \hat{\psi}_{j,\varepsilon} = f_j\chi_n \quad j = 1, 2.$$

Then

$$(5.24) \quad \left\| \left[\tilde{H}_{\delta,\varepsilon} - \frac{n^2\pi^2}{\delta^2} - z \right] \hat{\Psi}_\varepsilon - \tilde{\Xi}_n \right\|_{\tilde{\mathbb{H}}_\varepsilon} = \frac{1}{\varepsilon^{3/2}} \left\| \left[\tilde{L}_{\delta/\varepsilon} - \frac{n^2\pi^2}{(\delta/\varepsilon)^2} - \varepsilon^2 z \right] \hat{\psi}_{v,\varepsilon} \right\|_{L^2(V)}.$$

We notice moreover that from the definition of p_1 and p_2 , see equation (3.13), one has

$$(5.25) \quad |p_j| \leq c \|f_j\|_{L^2((0,\infty))}; \quad j = 1, 2.$$

Using the last estimate in equations (3.17) and (5.14) it follows that

$$[|\xi_{1,\varepsilon}| + |\xi_{2,\varepsilon}|] \leq c \|\tilde{\Xi}_n\|_{\tilde{\mathbb{H}}_\varepsilon}; \quad \left[\|\hat{\psi}_\varepsilon^*\|_{L^2(V)} + \left\| \frac{\partial}{\partial s} \hat{\psi}_\varepsilon^* \right\|_{L^2(V)} \right] \leq c \|\tilde{\Xi}_n\|_{\tilde{\mathbb{H}}_\varepsilon},$$

where we used $\|\tilde{\Xi}_n\|_{\tilde{\mathbb{H}}_\varepsilon} = [\|f_1\|_{L^2((0,\infty))}^2 + \|f_2\|_{L^2((0,\infty))}^2]^{1/2}$ and where c is a constant which does not depend on ε , f_1 , f_2 and n . Then Theorem 1 follows from the last estimate, from equation (5.24) and from Proposition 5.6. \square

6. LIMIT OPERATOR ON THE GRAPH

We denote by \mathbb{H}_G the complex Hilbert space

$$\mathbb{H}_G := L^2((0, \infty)) \oplus L^2((0, \infty))$$

with standard scalar product and norm. In \mathbb{H}_G we define the following selfadjoint operators:

Definition 6.1 (Limit operators on the graph).

$$D(\mathbf{h}^{dec}) := \{(x_1, x_2) \in \mathbb{H}_G | x_1, x_2 \in H^2((0, \infty)); x_1(0) = x_2(0) = 0\}$$

$$\mathbf{h}^{dec}(x_1, x_2) = (-x_1'', -x_2'').$$

$$(6.1) \quad D(\mathbf{h}^{\alpha_1\alpha_2}) := \left\{ (x_1, x_2) \in \mathbb{H}_G \mid x_1, x_2 \in H^2((0, \infty)); \Lambda_0^\perp \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = 0; \Lambda_0 \begin{pmatrix} x_1'(0) \\ x_2'(0) \end{pmatrix} = 0 \right\}$$

$$(6.2) \quad \mathbf{h}^{\alpha_1\alpha_2}(x_1, x_2) = (-x_1'', -x_2''),$$

where Λ_0 is the projector defined in equation (3.18).

We notice that the operators \mathbf{h}^{dec} and $\mathbf{h}^{\alpha_1\alpha_2}$ belong to the family of operators $-\Delta_{\mathcal{G}}^{\Pi,\Theta}$ described in the introduction. In particular the operator \mathbf{h}^{dec} coincides with the Dirichlet (or decoupling) Laplacian on the graph and the operator $\mathbf{h}^{\alpha_1\alpha_2}$ coincides with the weighted Laplacian (for $N = 2$).

For any $z \in \mathbb{C} \setminus \mathbb{R}$ we denote by $\mathbf{r}^{dec}(z)$ and $\mathbf{r}^{\alpha_1\alpha_2}(z)$ the resolvents of \mathbf{h}^{dec} and $\mathbf{h}^{\alpha_1\alpha_2}$ respectively, $\mathbf{r}^{dec}(z) = (\mathbf{h}^{dec} - z)^{-1}$ and $\mathbf{r}^{\alpha_1\alpha_2}(z) = (\mathbf{h}^{\alpha_1\alpha_2} - z)^{-1}$. We notice that for any vector $(f_1, f_2) \in \mathbb{H}_{\mathcal{G}}$

$$\mathbf{r}^{dec}(z)(f_1, f_2) = (r_0(z)f_1, r_0(z)f_2),$$

where the operator $r_0(z)$ was defined in equation (3.12). Moreover

$$\mathbf{r}^{\alpha_1\alpha_2}(z)(f_1, f_2) = (r_0(z)f_1 + q_1 e^{i\sqrt{z}}, r_0(z)f_2 + q_2 e^{i\sqrt{z}}),$$

with

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{i\Lambda_0}{\sqrt{z}} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}; \quad p_1 = (r_0(z)f_1)'(0); \quad p_2 := (r_0(z)f_2)'(0);$$

where p_1 and p_2 are defined accordingly to the definition used in equations (3.13).

Let us define the operator $\tilde{\mathbb{P}}_n : \tilde{\mathbb{H}}_{\varepsilon} \rightarrow \mathbb{H}_{\mathcal{G}}$

$$\tilde{\mathbb{P}}_n \tilde{\Psi} := ((\chi_n, \tilde{\psi}_1)_{L^2((0,1))}, (\chi_n, \tilde{\psi}_2)_{L^2((0,1))}) \in \mathbb{H}_{\mathcal{G}},$$

for any vector $\tilde{\Psi} \equiv (\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_v) \in \tilde{\mathbb{H}}_{\varepsilon}$. We denote by $\tilde{\mathbb{P}}_n^*$ the adjoint of $\tilde{\mathbb{P}}_n$, i.e., $\tilde{\mathbb{P}}_n^* : \mathbb{H}_{\mathcal{G}} \rightarrow \tilde{\mathbb{H}}_{\varepsilon}$ and for any vector $(g_1, g_2) \in \mathbb{H}_{\mathcal{G}}$

$$\tilde{\mathbb{P}}_n^*(g_1, g_2) = (g_1 \chi_n, g_2 \chi_n, 0) \in \tilde{\mathbb{H}}_{\varepsilon}.$$

For any vector $(g_1, g_2) \in \mathbb{H}_{\mathcal{G}}$ and any vector $\tilde{\Psi} \equiv (\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_v) \in \tilde{\mathbb{H}}_{\varepsilon}$ we have:

$$((g_1, g_2), \tilde{\mathbb{P}}_n \tilde{\Psi})_{\mathbb{H}_{\mathcal{G}}} = ((g_1 \chi_n, g_2 \chi_n, 0), \tilde{\Psi})_{\tilde{\mathbb{H}}_{\varepsilon}} = (\tilde{\mathbb{P}}_n^*(g_1, g_2), \tilde{\Psi})_{\tilde{\mathbb{H}}_{\varepsilon}}.$$

Theorem 3. For any vector $\tilde{\Xi}_n \equiv (f_1 \chi_n, f_2 \chi_n, 0)$, with $\|\tilde{\Xi}_n\|_{\tilde{\mathbb{H}}_{\varepsilon}} = 1$, and for all $z \in \mathbb{C} \setminus \mathbb{R}$ there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and for all $0 < \delta \leq \varepsilon$ the following estimates hold true:

Case 1.

$$\left\| \left[\tilde{\mathbb{P}}_n \tilde{R}_{\delta, \varepsilon}(z - n^2 \pi^2 / \delta^2) - \mathbf{r}^{dec}(z) \tilde{\mathbb{P}}_n \right] \tilde{\Xi}_n \right\|_{\mathbb{H}_{\mathcal{G}}} \leq c \frac{\delta}{\varepsilon^{3/2}};$$

Case 2.

$$\left\| \left[\tilde{\mathbb{P}}_n \tilde{R}_{\delta, \varepsilon}(z - n^2 \pi^2 / \delta^2) - \mathbf{r}^{\alpha_1\alpha_2}(z) \tilde{\mathbb{P}}_n \right] \tilde{\Xi}_n \right\|_{\mathbb{H}_{\mathcal{G}}} \leq c \left[\frac{\delta}{\varepsilon^{3/2}} + \frac{\delta}{\varepsilon^{5/2}} \right];$$

where c is a constant which does not depend on ε , f_1 , f_2 and n , and

$$\tilde{R}_{\delta, \varepsilon}(z - n^2 \pi^2 / \delta^2) := \left(\tilde{H}_{\delta, \varepsilon} - \frac{n^2 \pi^2}{\delta^2} - z \right)^{-1}.$$

Proof. We give the proof for the Case 2 only. The proof in the Case 1 is identical and we omit it. Since the operators $\tilde{R}(z - n^2 \pi^2 / \delta^2)$ and $\mathbf{r}^{\alpha_1\alpha_2}(z)$ are bounded by $|\operatorname{Im} z|^{-1}$, it is enough to prove that for any vector $(g_1, g_2) \in \mathbb{H}_{\mathcal{G}}$ one has

$$\left| \left((g_1, g_2), \left[\tilde{\mathbb{P}}_n \tilde{R}_{\delta, \varepsilon}(z - n^2 \pi^2 / \delta^2) - \mathbf{r}^{\alpha_1\alpha_2}(z) \tilde{\mathbb{P}}_n \right] \tilde{\Xi}_n \right)_{\mathbb{H}_{\mathcal{G}}} \right| \leq c \left(\frac{\delta}{\varepsilon^{3/2}} + \frac{\delta}{\varepsilon^{5/2}} \right).$$

The following inequality holds true

(6.3)

$$\begin{aligned} & \left| \left((g_1, g_2), \left[\tilde{\mathbb{P}}_n \tilde{R}_{\delta, \varepsilon}(z - n^2 \pi^2 / \delta^2) - \mathbf{r}^{\alpha_1\alpha_2}(z) \tilde{\mathbb{P}}_n \right] \tilde{\Xi}_n \right)_{\mathbb{H}_{\mathcal{G}}} \right| \\ & \leq \left| \left((g_1, g_2), \tilde{\mathbb{P}}_n \left[\tilde{R}_{\delta, \varepsilon}(z - n^2 \pi^2 / \delta^2) \tilde{\Xi}_n - \hat{\Psi}_{\varepsilon} \right] \right)_{\mathbb{H}_{\mathcal{G}}} \right| + \left| \left((g_1, g_2), \left[\tilde{\mathbb{P}}_n \hat{\Psi}_{\varepsilon} - \mathbf{r}^{\alpha_1\alpha_2}(z) \tilde{\mathbb{P}}_n \tilde{\Xi}_n \right] \right)_{\mathbb{H}_{\mathcal{G}}} \right|, \end{aligned}$$

where the vector $\hat{\Psi}_\varepsilon$ is the one given in Definition 3.1. From Remark 3.2 and from the Theorem 1 it follows that in *Case 2*

$$\begin{aligned} & \left| \left((g_1, g_2), \tilde{\mathbb{P}}_n \left[\tilde{R}_{\delta, \varepsilon}(z - n^2 \pi^2 / \delta^2) \tilde{\Xi}_n - \hat{\Psi}_\varepsilon \right] \right)_{\mathbb{H}_G} \right| \\ &= \left| \left(\tilde{\mathbb{P}}_n^*(g_1, g_2), \left[\tilde{R}_{\delta, \varepsilon}(z - n^2 \pi^2 / \delta^2) \tilde{\Xi}_n - \hat{\Psi}_\varepsilon \right] \right)_{\tilde{\mathbb{H}}_\varepsilon} \right| \leq c \left(\frac{\delta}{\varepsilon^{3/2}} + \frac{\delta}{\varepsilon^{5/2}} \right). \end{aligned}$$

The second term on the right hand side of equation (6.3) can be explicitly written as

$$\left| \left((g_1, g_2), \left[\tilde{\mathbb{P}}_n \hat{\Psi}_\varepsilon - \mathbf{r}^{\alpha_1 \alpha_2}(z) \tilde{\mathbb{P}}_n \tilde{\Xi}_n \right] \right)_{\mathbb{H}_G} \right| = \left| \left((g_1, g_2), ([q_{1, \varepsilon} - q_1] e^{i\sqrt{z} \cdot}, [q_{2, \varepsilon} - q_2] e^{i\sqrt{z} \cdot}) \right)_{\mathbb{H}_G} \right| \leq c,$$

where we used the result of Theorem 2, equation (3.16), and the estimate (5.25) (we recall that we are assuming $\|\tilde{\Xi}_n\|_{\tilde{\mathbb{H}}_\varepsilon} = 1$, then $\|f_j\|_{L^2((0, \infty))} \leq 1$, $j = 1, 2$). \square

7. CONCLUSIONS

We used the unitary map $U_{\delta, \varepsilon}$, see equation (2.7) and studied the problem in the Hilbert space $\tilde{\mathbb{H}}_\varepsilon$. In this section we discuss our results in the more natural Hilbert space $\mathbb{H}_{\delta, \varepsilon}$.

In our model the width of the waveguide δ must be intended as a function of ε such that $0 < \delta(\varepsilon) < \varepsilon^a$, for some big enough, positive constant a . In what follows we shall not mark explicitly the fact that δ depends on ε .

Let us define the functions $\chi_{\delta, n}$

$$\chi_{\delta, n}(u) := \delta^{-1/2} \chi_n(u) = \sqrt{\frac{2}{\delta}} \sin n\pi u \quad n = \mathbb{N},$$

where the functions χ_n were defined in equation (3.8).

The main idea of our approach is to find an approximate solution for the resolvent equation, i.e., for any vector in $\mathbb{H}_{\delta, \varepsilon}$ of the form $\Xi_{\delta, n} = (f_1 \chi_{\delta, n}, f_2 \chi_{\delta, n}, 0)$, where $f_1, f_2 \in L^2((0, \infty))$ we look for an approximate solution of the equation

$$(7.1) \quad \left[H_{\delta, \varepsilon} - \frac{n^2 \pi^2}{\delta^2} - z \right] \Psi = \Xi_{\delta, n} \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

In section 3 we showed an explicit approximate solution which is in the domain of the Hamiltonian $\tilde{H}_{\delta, \varepsilon}$. The following proposition is a consequence of the fact that $U_{\delta, \varepsilon}$ is unitary and maps $D(H_{\delta, \varepsilon})$ to $D(\tilde{H}_{\delta, \varepsilon})$, the proof follows directly from Theorem 1.

Proposition 7.1. *For any vector $\Xi_{\delta, n} \equiv (f_1 \chi_{\delta, n}, f_2 \chi_{\delta, n}, 0)$ with $f_1, f_2 \in L^2((0, \infty))$ let us take $\Psi_\varepsilon = U_{\delta, \varepsilon}^{-1} \hat{\Psi}_\varepsilon$, where the vector $\hat{\Psi}_\varepsilon$ was given in Definition 3.1 and $U_{\delta, \varepsilon}^{-1}$ is the inverse of the unitary map $U_{\delta, \varepsilon}$ defined in equation (2.7). Moreover in Case 1 assume $\delta(\varepsilon) < \varepsilon^{3/2}$ and in Case 2 assume $\delta(\varepsilon) < \varepsilon^{5/2}$. Then $\Psi_\varepsilon \in D(H_{\delta, \varepsilon})$, and for all $z \in \mathbb{C} \setminus \mathbb{R}$*

$$(7.2) \quad \left[H_{\delta, \varepsilon} - \frac{n^2 \pi^2}{\delta^2} - z \right] \Psi_\varepsilon = \Xi_{\delta, n} + \Phi_\varepsilon$$

with

$$\|\Phi_\varepsilon\|_{\mathbb{H}_{\delta, \varepsilon}} \leq c_\varepsilon \|\Xi_{\delta, n}\|_{\mathbb{H}_{\delta, \varepsilon}},$$

where c_ε does not depend on f_1, f_2 and n , and $c_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$.

To prove the convergence to an operator on the graph (in norm resolvent sense) we notice that in the edges the approximate solution Ψ_ε is factorized in the coordinates s and u . The transverse component is of the form $\chi_{\delta, n}$ and the longitudinal component can be written as it is done in the following equation (see also Definition 3.1):

$$(7.3) \quad \begin{aligned} \psi_{1, \varepsilon}(s, u) &= [(r_0(z) f_1)(s) + q_{1, \varepsilon} e^{i\sqrt{z}s}] \chi_{\delta, n}(u) \equiv x_{1, \varepsilon}(s) \chi_{\delta, n}(u) \\ \psi_{2, \varepsilon}(s, u) &= [(r_0(z) f_2)(s) + q_{2, \varepsilon} e^{i\sqrt{z}s}] \chi_{\delta, n}(u) \equiv x_{2, \varepsilon}(s) \chi_{\delta, n}(u), \end{aligned}$$

We interpret the s dependent parts in $\psi_{1,\varepsilon}$ and $\psi_{2,\varepsilon}$, i.e., the functions $x_{1,\varepsilon}$ and $x_{2,\varepsilon}$, as functions on the edges of the graph and consider the vector $(x_{1,\varepsilon}, x_{2,\varepsilon}) \in \mathbb{H}_G$. By definition for all $\varepsilon > 0$, we have $x_{1,\varepsilon}, x_{2,\varepsilon} \in H^2((0, \infty))$ and these functions satisfy the equation

$$-x''_{j,\varepsilon} - zx_{j,\varepsilon} = f_j \quad j = 1, 2.$$

As we are able to compute the limit of $x_{\varepsilon,1}(0) \equiv q_{1,\varepsilon}$, $x_{\varepsilon,2}(0) \equiv q_{2,\varepsilon}$, $x'_{\varepsilon,1}(0) \equiv \xi_{1,\varepsilon}$ and $x'_{\varepsilon,2}(0) \equiv \xi_{2,\varepsilon}$ as $\varepsilon \rightarrow 0$, we can prove that, see Theorem 2, in *Case 1*

$$\lim_{\varepsilon \rightarrow 0} x_{\varepsilon,1}(0) = 0, \quad \lim_{\varepsilon \rightarrow 0} x_{\varepsilon,2}(0) = 0, \quad \lim_{\varepsilon \rightarrow 0} x'_{\varepsilon,1}(0) = p_1, \quad \lim_{\varepsilon \rightarrow 0} x'_{\varepsilon,2}(0) = p_2,$$

implying that the limit operator on the graph is the Laplacian with Dirichlet conditions in the vertex;

while in *Case 2*

$$(7.4) \quad \lim_{\varepsilon \rightarrow 0} [\alpha_2 x_{\varepsilon,1}(0) - \alpha_1 x_{\varepsilon,2}(0)] = 0, \quad \lim_{\varepsilon \rightarrow 0} [\alpha_1 x'_{\varepsilon,1}(0) + \alpha_2 x'_{\varepsilon,2}(0)] = 0$$

where α_1 and α_2 are two real constants defined by the zero energy eigenvector y^* of the Hamiltonian h_v , see equations (3.6) and (3.7). In this case the limit operator on the graph is the Laplacian with a weighted Kirchhoff condition in the vertex. In *Case 2* one of the constants α_1 or α_2 may be equal to zero but they cannot be both equal to zero; in this case the limit operator on the graph is defined by a Dirichlet condition on one of the edges and a Neumann condition on the other.

We remark that the analysis performed here can be used also to prove the results of [6] and [7] which cover some generalizations of the model presented in [1].

We also notice that our method applies to settings with one or both edges having finite length. The form of the solution in the vertex region (i.e. the function $\hat{\psi}_{v,\varepsilon}$ in Definition 3.1) does not change, while the solution in the edge regions must be adapted to fulfill the boundary condition in each endpoint.

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